Rapid Evaluation of Coaxiality of Shaft Parts Based on Double Maximum Material Requirements

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Abstract: Simultaneous application of the maximum material requirements on the datum and coaxiality tolerances (referred to as DMMR coaxiality) of shaft parts can ensure assembly and reduce costs. Existing DMMR coaxiality evaluations use either inflexible real functional gauges or slow mathematical methods, which limits the applications of such a good tolerance in industry. This paper investigated a fast mathematical evaluation method for DMMR coaxiality. First, according to ISO requirements, the geometry and utility of the real functional gauge were analyzed. Then, an adaptive virtual gauge (AVG) was established, and the geometric structure and motion of the AVG were analyzed. After that, the mathematical evaluation method of the DMMR coaxiality tolerance was provided with uncertainty analysis. Finally, an exemplary application on a stepped shaft was presented, and the accuracy and speed of the method were reflected and improved by comparison with existing methods.

Keywords: Coaxiality; datum; mathematical model; maximum material requirement; rapid evaluation, uncertainty.

1 Introduction

The dimensional and coaxial errors of shaft products have an important impact on their performance; therefore, geometric evaluation is essential during and after manufacturing. In any evaluation method, dimensional and coaxial tolerances are two important types of information to describe the geometric qualities of a product. These two types of tolerances could compensate for each other when the maximum material requirement (MMR) is applied. Simultaneous application of MMRs on the datum and coaxiality tolerances facilitates higher part acceptance and ensures the assembly [1]. This tolerance is referred to herein as the coaxiality tolerance based on the double maximum material requirement (DMMR coaxiality tolerance). It is important to quickly and accurately evaluate DMMR coaxialities of shafts.

The DMMR coaxiality tolerance establishes two size-fixed virtual hole boundaries: one for the toleranced cylinder and the other for its datum cylinder. When an actual datum shaft is smaller than the maximum material size (MMS), its movement relative to datum reference (MRDR) is allowed, and more errors can be compensated [1]. This compensation is not linear, but a complex combination [9]. Therefore, the tolerance is mainly evaluated by real functional gauge, and the existing five types of mathematical evaluation methods have difficulties assessing such errors:

1. Direct calculation. As MRDR is allowed, the actual toleranced shaft will float with the actual datum shaft. Therefore, it is not possible to directly calculate the coaxiality.
2. The least squares method. This method is an approximate evaluation method based on “average error,” and is not suitable for DMMR error evaluation based on the “combined virtual boundary” [2][3].
3. The convex hull method. The convex hull method first establishes the convex hull of a single feature and then finds the minimum error of the convex hull by an enumeration [4]. However, like the direct calculation method, since the traditional error parameters cannot be directly calculated, the convex hull method can only be used to simplify the measurement data.
4. The optimization method. At present, optimization techniques are widely used in error evaluation, such as the simplex method [5], differential evolution algorithm [6][7], and sequential quadratic programming (SQP) [8]. Our team also tried to model the DMMR coaxiality error with the constrained target optimization problem and used several optimization algorithm to solve it [9]. Unfortunately, the classic optimization method is not efficient when used to evaluate the

geometric error of a large number of measuring points (they took more than 2 seconds to evaluate less than 2600 measurement points).

5. The geometric search method. When evaluating the corresponding features, the search direction of the geometric search method is obtained by geometric operations from a small number of measuring points at the boundary, and the efficiency and stability of the algorithm are usually high. This method is currently used for error evaluation of single type features, such as straightness error [10], sphericity of balls [11], and flatness of planes [12]. However, each geometric search algorithm is difficult to apply to other types of features, which makes it difficult to develop a new geometric search method for DMMR coaxiality.

In summary, when there is a large number of measuring points, the current five types of geometric error evaluation methods are either difficult or inefficient to apply to DMMR coaxiality. Therefore, this paper investigated a DMMR coaxiality error evaluation method that was derived from general laws and had high efficiency. First, an adaptive virtual gauge (AVG) was established by analyzing the geometry and function of the real functional gauge. Then, the geometry and motion process of the AVG were mathematically modeled. After that, a DMMR coaxiality error evaluation method based on the AVG was proposed with uncertainty analysis. Finally, a case study was conducted to illustrate the proposed method, compare it with classic methods, and improve the proposed method.

2 Constraints and real gauge of DMMR coaxiality

The DMMR coaxiality tolerance of the shaft has the five geometric requirements [1]. The first one is about sizes: **Requirement 1:** The local sizes of the shafts are between their maximum material size (MMS) and their least material size (LMS). The other four describe the geometry of two coaxial boundaries with their own sizes. **Requirement 2:** The shafts must not violate their maximum material virtual boundaries (MMVBs). **Requirement 3:** The MMVBs of the tolerated shaft and its datum shaft are coaxial. **Requirement 4:** The size of the MMVB — the maximum material virtual size (MMVS) — of the tolerated shaft is equal to its MMS plus the coaxiality tolerance value. **Requirement 5:** When the datum shaft is marked with form tolerance and the form tolerance has no MMR, the MMVS of the datum shaft is its MMS; when the datum shaft is marked with a form tolerance and the form tolerance has MMR, the MMVS of the datum shaft is equal to its MMS plus the form tolerance.

When **Requirement 1** is met, the DMMR coaxiality of the shaft can be verified with a real functional gauge [1]. An example was given in [9]: the DMMR coaxiality of the shaft shown in Figure 1(a) can be verified by the integrated gauge shown in Figure 1(b). The gauge consists of two coaxial holes, which are denoted as A_G and b_G . The diameter of hole b_G is the MMVS of the tolerated shaft b ($\text{MMVS} = D_b + t_3 + T_b$). The diameter of hole A_G is the MMVS of the datum shaft A ($\text{MMVS} = D_A + t_1 + T_A$). The part being tested must pass the gauge.

Figure 1. (a) Drawing and (b) real gauge based verification of a DMMR coaxiality tolerance [9].

The method covers **Requirements 2–5**, but the real gauge based verification is costly and less flexible in small and medium batch production because the gauge and product specifications have one-to-one correspondence. In addition, virtual parts cannot be tested in real gauges. In order to improve the flexibility and evaluate the virtual parts, an AVG-based mathematical method corresponding to the real gauge-based method was established.

3 Geometry and motion analysis

In this section, a proper AVG is provided, and its key factors are analyzed. We start with representing the real shaft by its measurements, providing an AVG, and introducing their position, shape, size and movement. Then, we describe the geometry and motion of the measurements and the AVG, and establish the basic motion equations of contact points. After that, we analyze the motion equations to reveal the motion (or motionless) states of the measurements and the boundaries, by identifying virtual constraints, virtual contacts and over-constraints ([Figure 2](#)). We also provide how to find out a new contact point (if AVG continues shrinking).

Figure 2. An AVG and its application. The shrinkage process is that: (a), (b) SCD shrinks, (c) SCD remains unchanged, SCT shrinks, and (d) SCT shrinks to the minimum.

3.1 Geometry and motion process

In this subsection, we first introduce actual part's measurements, a general AVG and the checking process. Then, we

discussed proper and necessary assumptions and simplification (without changing the ‘reading’), before the simplified AVG and its use are detailed.

3.1.1 General AVG and its use

Real shafts and its simulating (virtual shafts) are called actual shafts. In this paper, the measurements of an actual shaft are coordinates in a Cartesian coordinate system, and the acquisition method can be found in [13]. The measurements can be verified in a virtual gauge, which is a concept that can be analyzed using mathematical tools. For the DMMR coaxiality error evaluation, the AVG constructed in this paper has two coaxial shrinking boundaries: the shrinking cylinder for datum shaft (SCD) and the shrinking cylinder for the toleranced shaft (SCT), as shown in Figure 2. To meet the five **requirements**, the SCD first attempts to shrink and push the datum measurements into its MMVB. After that, the SCD remains unchanged and keeps the datum measurements inside, while the SCT attempts to shrink and push the measurements. Under the combined effect, the SCT eventually shrinks to the limit equivalent size. If the size is not greater than toleranced shaft’s MMVS, the part is supposed to pass the AVG (and a real gauge).

3.1.2 Assumptions and simplification

The general AVG and its use are ready in concept to check measurements of an actual gauge, but not clear enough for calculation. Thus, we should make some preparations to derive a simple and clear shrinking, contacting and pushing for calculation, including identifying reading-unrelated factors and preventing them from complicating the process, ensuring that the contact points touch the gauge one by one, and assuming (and proving it in Section 3.2) the validity of contact points between each touch. (The ‘touch’ means that a measurement point was not a contact point, but it approaches and tries to pass the AVG at the current time.)

First, if a part is checked several times by the shrinking gauge as in Section 3.1.1, the detailed process may be different each time while the ‘readings’ should be the same. Intuitively, the structure of the gauge and the surface of the part decide the ‘reading’, while the location, speed and force (if gently enough) only make the process complex. Thus, in order to get the correct ‘reading’ in a simpler process, it is necessary to ensure the structure of the gauge and keep the surface rigid, while helpful to fix the common axis of the AVG to an axis (i.e. the z axis), ‘select a shrinking speed’ and ignore the force. Therefore, we make **Assumption 1**.

Assumption 1: The common axis of the SCD and SCT are fixed to the z axis, shrink at a constant speed, pushes and blocks the contact points gently, and do not separate from them, while the measurement points move together as a rigid body.

Second, it is sure to be easier to analyze the motion of contact points, if the measurements contact the gauge one by one. Thus, it is useful to make a simplification to decide a unique new contact point in complex cases. The first complex case is when there are multiple points touching the AVG at the same time. In this case, we randomly select one of them as the new contact point; in this way, we can make sure whether the others are going to touch the AVG at the next moment. Another complex case is when the SCD gets the datum shaft’s MMVS and the SCT start to shrinking. At this time, all of the old contact points are on the SCD, as the SCD stops shrinking. Thus, we have to choose a new contact point on the SCT to keep pushing the measurements. All in all, we make **Simplification 1**.

Simplification 1: When there are multiple points touching the AVG at the same time, randomly select one of them as the new contact point; when the SCD shrinks to the datum shaft MMVS, select the new contact point on the SCT.

Third and the last, even the measurements contact the AVG boundaries here or there time and time during the evaluating, virtual constraints and virtual contact point will leave the AVG after each touch, until an over-constraint occurs, as shown in [Figure 2](#). Thus, we make **Assumption 2**, and discuss when it is true in [Section 3.2](#).

Assumption 2: After the last $((k-1)$ -th) and before the current (k) -th touch, the contact points are physical contact points, and no virtual constraints, virtual contacts, or over-constraints are formed.

The main checking process is that the AVG shrinking, touching, and pushing the contact points over and over again. After the reasonable simplification and assumptions make them clear and simple, we believe that the AVG and its use are ready for calculation.

3.1.3 Details of simplified AVG and its use

According to **Assumption 1**, **2** and **Simplification 1**, the simplified AVG could be created and used as followed.

First of all, the SCD should be created at the z -axis and contain datum measurements, as shown in [Figure 2\(a\)](#). Therefore, if the datum shaft exceeds its MMVB, the radius of the SCD should be the maximum distance from the datum measurements to the z -axis, while the corresponding point should be the first contact point; otherwise, the SCD size should be the datum shaft's MMVS.

If the SCD is larger than the datum shaft's MMVB, the SCD attempts to shrink to the MMVS and pushes the measurement points at the physical contact points, as shown in [Figures 2\(a\) through 2\(c\)](#). Even the AVG shrinks at a constant speed, the motion (translation and rotation) of the measurement points often changes and always depends on the current physical contact points. However, not all contact points are physical. At the moment of a touch, it may create (i) virtual constraints; (ii) virtual contact points ([Figure 2\(c\)](#)); or (iii) over-constraints ([Figure 2\(d\)](#)). In contrast of physical contact point, a virtual constraint is redundant (and will not exist, see [Appendix A](#)), a virtual contact point will leave the AVG after an instant, and an over-constraint will stop the AVG from shrinking; thus they will not exist between two touches (that is, **Assumption 2** is true).

Once the SCD has been already at the datum shaft's MMVB, the SCT should be constructed. To contain the toleranced shaft, the initial radius of the SCT should be the maximum distance from the toleranced shaft to the z -axis at that time, and the corresponding point should be the new contact point. Then, the SCT attempts to shrink, while the SCD remains unchanged and blocks the datum shaft measurements. The motion of the measurement points depends on the combined effect, as shown in [Figures 2\(c\) and 2\(d\)](#). In the process, new contacts, virtual contacts, virtual constraints, and over-constraints may also occur.

No matter when over-constraints occur, the AVG stops shrinking. If the it stops (i) before the SCD shrinks to the datum shaft's MMVS, the actual datum shaft is proved too large (not meet the size **Requirement 1**); (ii) otherwise, the measurement points stop and the SCT is at its minimum size—the limit equivalent size ('reading'). If the limit equivalent size is not greater than the MMVS of the toleranced shaft, the part is considered to meet **Requirements 2–5**; otherwise, it is not. Of cause, the uncertainty of the result should be also evaluated.

3.2 Description of geometry and motion

When an AVG is used as simplified, it shrinks at a constant speed, but the contact points (and other measurement points) do not move constantly. Since the AVG keeps contacting and pushing the measurements at the physical contact points, we analyze the contact points and establish the basic motion equation of them, after describing the geometry and motion of the AVG and the measurements, respectively, as shown in [Figure 3](#).

Figure 3. Geometry and motion of measurements and AVG: (a) Measurement set, (b) a single measurement point, (c) a point of the shrink boundary, and (d) the position and velocity of a contact point.

On the AVG, the boundary point $P_i = (X_i, Y_i, Z_i)^T$ is possible to shrink and push the measurement point $p_i = (x_i, y_i, z_i)^T$, as shown in Figure 3(c). If the AVG shrinks, P_i moves toward the axis of the AVG. Thus, P_i 's velocity direction (as the same as the shrinking direction) is $N_i = (-X_i/R_{CD}, -Y_i/R_{CD}, 0)^T = (-x_i/r_i, -y_i/r_i, 0)^T$, and its speed is $V_i = b_i N_i$; where (i) b_i is P_i 's projection of V_i in the shrinking direction (uniform speed), and its value is 0 (when the boundary keeps stationary) or a positive constant b (when the boundary is shrinking); (ii) R_{CD} is the radius of the boundary; and (iii) r_i is the distance from p_i to the z -axis and is calculated as Equation (1).

$$r_i = (x_i^2 + y_i^2)^{0.5}. \quad (1)$$

The measurement set, as a rigid body, has up to six degrees of freedom in three-dimensional space: translation along the x , y , and z axes, denoted as dx , dy , and dz ; and rotation around the x , y , and z axes, denoted as drx , dry , and drz . For the stepped shaft, dz and drz do not change the AVG and the set of contact points. Therefore, as shown in Figure 3(a), the measurement points are assumed to not move by dz or drz . Thus, as shown in Figure 3(b), measurement point p_i 's velocity $v_i = (dx_i, dy_i, dz_i)^T$ can be calculated by Equation (2), according to [12].

$$v_i = T + R p_i, \quad (2)$$

where, $i \in I = \{1, 2, 3, \dots\}$, and $T = (dx, dy, 0)^T$. Additionally, as reflections of error, drx and dry are so small that their sine values are about the same as their own values and their product is about zero. Therefore,

$$R = \begin{bmatrix} 0 & 0 & dry \\ 0 & 0 & -drx \\ -dry & drx & 0 \end{bmatrix}.$$

As the AVG keeps contacting with anyone of the contact points, their projection of velocity in the shrinking direction N_i are the same, as shown in Figure 3(d). Marking the projection of velocity v_i of measurement point p_i as its normal linear velocity u_i , it can be obtain the u_i by Equation (3).

$$u_i = v_i \cdot N_i = (-x_i dx - y_i dy - x_i z_i dry + y_i z_i drx) / r_i. \quad (3)$$

Equation (3) is a linear expression that can be rewritten as Equation (4):

$$A_i \Psi = u_i, \quad (4)$$

where (i) $A_i = [-x_i/r_i, -y_i/r_i, y_i z_i/r_i, -x_i z_i/r_i]$ is the motion row vector of p_i , and (ii) $\Psi = [dx, dy, drx, dry]^T$ is the motion vector of the measurement set. In addition, in general, the position change of measurement point p_i is so small that the motion row vector A_i is considered unchanged during the evaluation.

If p_i is a physical contact, its normal linear velocity u_i is equal to the uniform speed b_i of P_i , as shown in Figure 3(d). Substituting $b_i = u_i$ into Equation (4), the basic motion equation of the contact point can be established as Equation (5).

$$A_i \Psi = b_i. \quad (5)$$

For all current (k -th) contact points, by substituting the current contact point set Q into Equation (5), the basic motion equation group can be obtained as Equation (6).

$$A \Psi = b, \quad (6)$$

where $A = [..., A_i^T, ...]^T$ is the motion coefficient matrix, $b = [..., b_i, ...]^T$ is the motion constant term of the contact point set, and $i \in Q$.

3.3 Motion analysis of contact point set

Virtual constraints, virtual contacts, and over-constraints may occur, when an AVG shrinks as simplified. Therefore, it is necessary to identify them and remove the virtual constraints and contacts. This can be done by the analysis of the basic motion equation group and ‘reduced motion equations’.

If there are Q current contact points, the basic motion equation group with Q equations can be established by Equation (6), and there will be three possible states of the ranks of the motion coefficient matrix A and the augmented matrix $(A|b)$: (1) **State 1**: $\text{rank}(A) = \text{rank}([A|b]) = Q-1 \leq 4$, (2) **State 2**: $\text{rank}(A) = \text{rank}([A|b]) = Q \leq 4$, and (3) **State 3**: $Q-1 = \text{rank}(A) < Q = \text{rank}([A|b]) \leq 5$. **State 1** indicates that a basic motion equation and a contact point are redundant, and a virtual constraint exists. **State 2** indicates that all contact points are physical contact points and the motion can be solved directly. And, **state 3** indicates that the basic motion equation group can not be solved directly and virtual contacts or over-constraints occur. In **state 3**, when we are trying to make sure whether a contact point p_i is virtual or not, we assume it’s virtual and check whether it will leave the boundary. For checking, the one-reduced equation group will be established by substituting the reduced contact point set (physical contact point set, as assumed) into Equation (6), as **Definition 1**.

Definition 1: When the motion row vectors of several contacts $\{p_i | i \in S \subset Q\}$ are removed from the motion coefficient matrix, the reduced equation group $\bar{A}_S \Psi = \bar{b}_S$ would be obtained, where $\bar{A}_S = \{[\cdots A_i^T \cdots]^T | i \in Q, i \notin S\}$ is the reduced coefficient matrix and $\bar{b}_S = \{[\cdots b_i^T \cdots]^T | i \in Q, i \notin S\}$ is the reduced constant term. Specifically, if only one contact S is removed, a one-reduced equation group $\bar{A}_S \Psi = \bar{b}_S$, a one-reduced coefficient matrix $\bar{A}_S$ and a one-reduced constant term $\bar{b}_S$ are obtained.

When there is a single virtual contact point p_i , the AVG shall push the physical contact points (the rest of the current contact points), and virtual contact point p_i will go inside the boundary after the instance. With a one-reduced equation, this can be described as Formula (7).

$$\exists \Psi : \bar{A}_i \Psi = \bar{b}_i, \text{ and } A_i \Psi > b_i \quad (7)$$

Further more, by a mathematical analysis in Appendix A, it is proved that, (i) virtual constraints (**state 1**) will not occur, (ii) if one or more virtual contact points exist (**state 3**), a single virtual contact point is sure to be found by one-reduced equation groups, and (iii) if it is in **state 3** and no single virtual points are found by one-reduced equation groups, then an over-constraint occurs, as shown in Figure 4. As a result, the major checking process is created as a loop that ‘find a new contact, analyze the ranks, remove a virtual contact point (if any), and solve motion equations’, and the loop will end with an over-constraint, again, as shown in Figure 4. Obviously, at the beginning and in the process of the major loop, between two touches, where are no virtual constraints, virtual contact points or over-constraints, as **Assumption 2**.

Figure 4. Analysis of the motion of the contact point set.

3.4 Move and contact of measurements

If the current contact has been analyzed and no over-constraints are found, the AVG will keep shrinking, at least until the next touch (a new contact point is generated). Meanwhile, when the SCD touches (shrinks to and gets) the datum shaft's MMVB, the SCD will stop shrinking and a new contact point should be generated at the SCT (**Simplified 1**). Obviously, whether a measurement point touch the AVG or the SCD touches its MMVB, the contact point set and the motion of the measurements will change. Therefore, we have to analyze the both kinds of potential touch, before we look for the new contact point.

After the current touch, if the AVG keeps shrinking and a non-contact point p_i 's normal linear velocity u_i is smaller than its corresponding uniform speed b_i , the point p_i 'shrinks' slower than the AVG and is expected to take time τ_i to touch the AVG, according to the relative distance and speed along the shrinking direction, as [Formula \(8\)](#).

$$\tau_i = (R_{CD} - r_i) / (b_i - u_i), \quad (8)$$

where R_{CD} is the radius of corresponding SCD or SCT, at the current (k -th) touch.

Meanwhile, if the SCD is shrinking, the gap between the SCD and MMVB will be smaller and smaller. Also, it will take time τ_D to touch the MMVB, according to the current gap and the shrinking speed, as [Formula \(9\)](#), where R_{DM} is the radius of the datum shaft's MMVB.

$$\tau_D = (R_{CD} - R_{DM}) / b, \quad (9)$$

Obviously, only the earliest coming touch can be achieved, and the duration between the current (k -th) and the next ($(k+1)$ -th) touch can be obtained by (i) [Formula \(10\)](#), when the SCD is larger than the MMVB and shrinking, or (ii) [Formula \(11\)](#) when the SCD has been its MMVB and the SCT is shrinking.

$$\tau = \min(\tau_i, \tau_D). \quad (10)$$

$$\tau = \min \tau_i. \quad (11)$$

According to [Formula \(8\)](#) and [\(9\)](#), the radials of the boundaries R_{CD} and the distances of measurements r_i are important to decide the new contact point, and varies during the shrinking. Thus, we should update them after each touch (time node). As the boundaries shrinks directly to the z -axis and the measurement points move along their corresponding shrinking direction, [Formula \(12\)](#) and [Formula \(13\)](#) could be employed, where the symbol ' $\leftarrow$ ' means the assigning.

$$r_i \leftarrow r_i - \tau u_i, \quad (12)$$

$$R_{CD} \leftarrow R_{CD} - \tau b_i. \quad (13)$$

4 AVG based error evaluation

In this section, flow chat and steps of the AVG based evaluation are provided. As the complete result includes the evaluation result and the (expand) uncertainty, we also discussed the uncertainty of the method.

4.1 Flow chat and steps of the AVG based evaluation

According to the geometry and motion model of the AVG and the measurement points, a mathematical method for DMMR coaxial error evaluation based on AVG is organized, which consists of seven steps, as shown in Figure 5. Intuitively, before evaluation, the measurement points could be pre-positioned to z-axis to reduce the coordinate value and the evaluation error. Note that the position is approximate and can be flexibly fitted with the principle of minimum area, minimum outer cut, maximum inscribed, least squares, minimum mean, and minimum limits.

Figure 5. Steps of AVG based evaluation method.

Step 1: Take the datum shaft measurements $\{p_i | i=1, 2, \dots, N\}$ as the active set, and do not add the tolerated shaft measurement points $\{p_i | i=N+1, 2, \dots, N+M\}$ to the active set (first, only the SCD is shrinking). Then, generate the motion row vector set $\{A_i\}$ for all the measurements according to Equation (4), and generate the uniform speed set for the datum shaft $\{b_i | i=1, 2, \dots, N; b_i=b>0\}$ according to Section 3.2. Based on Equation (1), calculate the distance set $\{r_i\}$ for all the measurements, and the radius of SCD is $R_D = \max r_i | i=1, 2, \dots, N$. After that, perform Sub-step 1.1.

Sub-step 1.1: Determine if the measurements of the tolerated shaft should be added to the active set. Calculate the MMVS and R_{DM} of the datum shaft according to Requirement 5. If $R_D \leq R_{DM}$, and the tolerated shaft measurements is not yet

added to the active set, proceed to *Sub-step 1.2*. Otherwise, go to Step 2.

Sub-step 1.2: Add the toleranced shaft measurements to the active set (the SCT is created), update the uniform speed set to $\{b_i | i=1, 2, \dots, N, b_i=0; i=N+1, 2, \dots, N+M, b_i=b>0\}$ (the SCT shrinks while the SCD does not), and update the value of R_D to R_{DM} . Then, perform Step 2.

Step 2: Add a new contact p_q to the original contact set such that the current contact set is $\{p_i | i \in Q \subseteq I\}$. (i) If Step 2 is performed for the first time and *Sub-step 1.2* is not performed (the SCD is larger than its MMVB and the first contact point is going to be found on the SCD), the point corresponding to the maximum value ($r_{\max D}$) of the datum shaft's distance set $\{r_i | i=1, 2, \dots, N\}$ is taken as a new contact. (ii) When Step 2 is performed for the first time after *Sub-step 1.2* (the SCT is just constructed and start shrinking), the point corresponding to the maximum value ($r_{\max C}$) of the toleranced shaft's distance set $\{r_i | i=N+1, 2, \dots, N+M\}$ is taken as a new contact. (iii) Otherwise, a common new contact points generated in Step 6 should be accepted. After that, perform Step 3.

Step 3: The motion coefficient matrix A and the motion constant term b are established according to the contact set $\{p_i | i \in Q\}$ and Equation (6). Afterward, perform Step 4.

Step 4: Rank analysis is performed on the motion coefficient matrix A and its augmented matrix $([A|b])$. Then,

(i) If $\text{rank}(A) = \text{rank}([A|b])$ (**State 2**), skip to **Step 5** directly.

(ii) If $\text{rank}(A) < \text{rank}([A|b])$ (**State 3**), solve the one-reduced equation groups: $\bar{A}_i \Psi = \bar{b}_i$, $i \in Q$. If a solution satisfies Equation (7), the corresponding point could be selected as the virtual contact point. Then, remove corresponding elements from the contact set Q , the motion coefficient matrix A , and the motion constant term b , and perform Step 5. If any of the one-reduced equation groups has no solution that satisfies Equation (7), there is an over-constraint that leads to Step 7.

Step 5: Solve the basic motion equation group $A\Psi=b$ (Step 3) and perform Step 6.

Step 6: Search the new contact point and update the distance set. First, calculate the normal linear velocity $u_i = A_i \Psi$ of all points according to Equation (4). Then, check each point in the active set and calculate the duration τ between the two touches according to Equations (8)–(10) (*Sub-step 1.2* is not performed) or Equations (8), (11) (*Sub-step 1.2* has been performed). Note that, just evaluate the expected duration for the points whose u_i is smaller than its b_i . The corresponding point will be taken as the new contact, unless the minimum expected duration is τ_D (In this case, the SCD touches its MMVB and the new contact point will be generated on the SCT in the coming step 2). After that, update all the distances $r_i = r_i - \tau u_i$ $i=1, 2, \dots, N+M$ according to Equation (12), and update $R_D = R_D - \tau b_i$ $i=1, 2, \dots, N$ and $R_C = R_C - \tau b_i$ $i=N+1, N+2, \dots, N+M$ according to Equation (13). Then, go to *Sub-step 1.1*.

Step 7: Calculate the geometric error of the parts and determine the compliance. If *Sub-step 1.2* is performed and the toleranced shaft's limit equivalent size $D_C=2R_C$ is surely not greater than its MMVS ($D_C+U \leq \text{MMVS}$, where U is the expand uncertainty of result), the part conforms to the specified DMMR coaxiality tolerance; otherwise, it does not.

4.2 Accuracy analysis

During the verification, there are many factors which generate errors and uncertainties, such as environment, measurement equipment, measuring procedure and calculation. Common uncertainty components are recommended to evaluated as in [14] and [15], while there are still two components that we are interested in. First, while we focus in the measurement points' movements along their normal, we ignore the boundary expansion caused by the measurement points' movements along their tangents. When the tangential movements are not small enough, notable deviation will be produced. Second, as the gap between the datum shaft and the MMVB supplies error compensation for toleranced shaft, the uncertainty of the datum shaft's measurements will increase the uncertainty of the toleranced shaft's evaluation result.

4.2.1 Uncertainty caused by deviation from shrinking direction

Figure 6(a) shows the worst case of the difference between a calculated boundary and the actual resultant boundary. Considering measurement points' movements in their normal directions, we obtain the calculated boundary by the AVG based

approach, and measurement point p_j finally moves to the calculated boundary. However, the movement may cause another point p_i to move along its tangent, and a deviation of the result is generated. If p_i was on the calculated boundary and moved along its tangent, the worst case will occur and the deviation of result gets its maximum. Thus, the uncertainty caused by the deviation (u_{Dev}) could be represented by the maximum deviation and obtained by Formula (14), as shown in Figure 6(a).

$$u_{Dev} = \left(\left[\begin{matrix} \max(r_i) \\ 0 \end{matrix} \right] + \left[\begin{matrix} 0 \\ dx + \max(|z_i|)dy \\ dy - \max(|z_i|)dx \end{matrix} \right] \right) - \max(r_i) \quad (14)$$

Furthermore, assuming that the major components of the combined standard uncertainty are uncertainty of measurement equipment (u_E) and the evaluation method, we can calculate the uncertainty of the measurements' final positions by Formula (15) and (16), where u_D and u_T are this type of uncertainties of the datum shaft and the tolerated shaft, respectively. Note that, the uncertainty component of measurement is half of its error limit [15].

$$u_D = \sqrt{u_E^2 + u_{Dev}^2} \quad i=1,2,\dots,N \quad (15)$$

$$u_T = \sqrt{u_E^2 + u_{Dev}^2} \quad i=N+1,N+2,\dots,N+M \quad (16)$$

Figure 6. Uncertainties components. (a) Uncertainty generated by tangential moving, and (b) uncertainty of shafts, (c) translation uncertainties, (d) possible rotation uncertainties, (e) allowed rotation uncertainties, and (f) allowed rotation uncertainties in the worst case.

4.2.2 Uncertainty combination

Also, the uncertainty of the datum will produce an uncertainty component of the evaluation result. As shown in Figure 6 (b), as the true datum shaft may be larger or smaller than the measurements, there would be a smaller or greater gap to reduce or increase compensation for the tolerance shaft error, in the form of translation and rotation.

On the one hand, if the shafts translate, the uncertainty of the datum shaft will result in the same amount of uncertainty of

(the datum shaft's and) the tolerance shaft's translation, as shown in Figure 6 (c). That is, the translation resultant uncertainty $u_{D,T}$ (of the tolerated shaft, and introduced from the datum shaft) should be the same as the uncertainty (u_D) of datum measurements' final positions, respectively.

On the other hand, if the shafts rotate, the increase or decrease of the gap will result in the increase or decrease of (the datum shaft's and) the tolerance shaft's rotation, as shown in Figure 6 (d), (e) and (f). According to the uncertainty (u_D) of the datum measurements' final positions, the maximum rotational deviation $u_{R,1}$ depends on the datum's measurement points on the plane XOY and can be calculated by Formula (17), in which case a true point of the datum exceeds its MMVB and should be pushed in, as shown in Figure 6 (d). However, a rotation is not allowed if any true point of the datum exceeds its MMVB accordingly, as shown in Figure 6 (e), where r_i is the distance of p_i 's final calculated position. In the worst case, at least one true point may be inside the datum's MMVB, but it is sure to exceed the boundary if there is more rotation. To avoid the exceeding, we adopted the maximum gap and the maximum distance to plane XOY to calculate the limit of allowed rotation, as shown in Figure 6 (f). Thus, the allowed rotational deviation $u_{R,2}$ can be calculated by Formula (18).

$$u_{R,1} = \sqrt{\frac{2u_D}{R_{DM}}} \quad (17)$$

$$u_{R,2} = (R_{DM} - \min(r_i) + u_D) / \max(|z_i|) \Big|_{i=1,2,\dots,N} \quad (18)$$

Obviously, the smaller one between the maximum rotational deviation $u_{R,1}$ and the allowed rotational deviation $u_{R,2}$ of the datum shaft could be combined with the tolerated shaft. In addition, the maximum resultant deviation on the tolerated shaft also relates to the maximum distance from the tolerated shaft to the XOY plane, as shown in Figure 6 (f). Thus, the rotation resultant uncertainty $u_{D,R}$ can be calculated by Formula (19).

$$u_{D,R} = \max(|z_i|) \min(u_{R,1}, u_{R,2}) \Big|_{i=N+1, N+2, \dots, N+M} \quad (19)$$

Furthermore, since the SCT attempts to shrink, the combination of the translation and rotation resultant uncertainties should be the smaller one. Thus, the resultant uncertainty U_D (of the tolerated shaft, and introduced from the datum shaft) can be calculated by Formula (20).

$$U_D = \min(u_{D,R}, u_{D,T}) \quad (20)$$

According to [14] and [15], the uncertainty of the tolerated shaft (u_T) and the uncertainty caused by the datum (U_D) should be combined and could be expanded by Formula (21), where U is the expanded uncertainty. Only when the sum of the result and the uncertainty is smaller than the tolerated shaft's MMVB, the conformity is proved.

$$U = 2\sqrt{U_D^2 + u_T^2} \quad (21)$$

5 Case verification and comparative analysis

5.1 AVG-based error evaluation

To evaluate the proposed and existing methods on actual parts, the tolerance and measurements of the stepped shaft in [9] are introduced:

The tolerance and measured data (on coordinate measuring machine) are introduced from [9], as shown in Figure 7 and 8, respectively.

The unit of measurement is millimeters, the angle unit is radians, and the time unit is seconds.

The coordinate measuring machine was GLOBAL CLASSIC SR 07. 10. 07 produced by Hexagon Metrology (Qingdao, Shandong, China) with a non-contact measurement error limit of 0.02 mm.

Non-contact scanning of actual parts was performed from three directions: angle A (around x-axis) = 60°, angle B (around

z -axis) = 0° ; angle A = -60° , angle B = 60° ; and angle A = -60° , angle B = -60° . The spot spacing and strip spacing were both 2 mm.

Figure 7. DMMR coaxiality tolerance for a stepped shaft [9].

Figure 8. Measurements of a stepped shaft [9].

The measurements of the datum shaft were $\{p_{D,ex,n}|n = 1, 2, \dots, 2093\}$, and the measurements of the tolerated shaft were $\{p_{C,ex,m}|m = 2094, 2095, \dots, 2541\}$. Before evaluation, the measurement points were pre-positioned according to Equation (22) (Least square, LSQ, based) to reduce the coordinate value and to reduce the evaluation error, where $\{p_{D,n}\}$ and $\{p_{C,m}\}$ were pre-positioned measurements for the datum shaft and the tolerated shaft, respectively.

$$\begin{cases} p_{C,m} = p_{C,ex,m} - \frac{1}{2093} \sum_{i=1}^{2093} p_{D,ex,i} \\ p_{D,n} = p_{D,ex,n} - \frac{1}{2093} \sum_{i=1}^{2093} p_{D,ex,i} \end{cases} \quad (22)$$

The AVG based evaluation was implemented in MATLAB (R2014a) through our custom program. Twenty-two contacts in total were searched in $\tau_E = 0.21$ seconds, and the tolerated shaft's limit equivalent size D_C was 14.9869 mm, which was smaller than its MMVS (15.04 mm). Meanwhile, as the expanded uncertainty of the result U was 0.0424 mm, the sum of the result and the uncertainty was 14.9869 mm + 0.0424 mm = 15.0293 mm, which was smaller than its MMVS (15.04 mm). Thus, the part was proved acceptable.

5.2 Comparative analysis

As mentioned in Section 1, our group developed another method to evaluate the DMMR coaxiality based on optimization techniques in a previous paper [9]. The solution and calculation time of the AVG-based method were compared to those of several optimization methods, to evaluate the accuracy and efficiency of the AVG-based method.

Based on the optimization model, three classic methods were also employed to determine the limit equivalent size in MATLAB (R2014a). Their performances are shown in Table 1: the *interior point* function output the size D_C of 15.0062 mm in $\tau_E = 2.55$ seconds; the *active-set* function output the size D_C of 15.0061 mm in $\tau_E = 3.25$ seconds; and the *sqp* function (sequence quadratic method) output the size D_C of 15.0061 mm in $\tau_E = 5.94$ seconds. In addition, the uncertainties of the results were calculated as in Section 4.2, while the uncertainties derived from tangential movements were be include. Thus, the uncertainties of the results U were all 0.0283 mm.

As shown in Table 1, the smallest result and uncertainty among the classic methods (15.0061 mm and 0.0283 mm by the active set method) was taken as the standard value (D_C^*) and standard uncertainty (U^*), the shortest calculation time among the classic methods (2.55 seconds by the interior point method) was taken as the standard time (τ_E^*). The differences between the standard value and the results of other classic methods were less than 0.0001 mm, indicating a high accuracy and consistency of all the classic methods. Additionally, the time costs of the other classic methods were less than 2.3 times of the standard time, showing similarity between the methods. Thus, it was ensured that the standard value, uncertainty and calculation time were proper to be the references.

In contrast, the AVG-base method provided the limit equivalent size of 14.9869 mm, which was 0.0192 mm smaller than the standard value. Also, the uncertainty of the result (0.0424mm) was 0.0141 mm greater than the standard uncertainty (0.0283 mm). Additionally, the AVG-base method used 0.21 seconds for calculation, which was 8% of the standard time. Thus the AVG-based method was more efficient but less accurate than the classic methods.

The lower accuracy of the AVG-based method was mainly because the motion row vectors A_i were constructed according to the measurements' original p_i values rather than their real-time positions. During the adjustment, the positions to structure the motion row vectors A_i will change slightly. The comparison shows that this small change has a negligible effect on the accuracy of the results.

Table. 1: Results and time obtained by various solution methods.

Method	D_C (mm)	$D_C - D_C^*$ (mm)	U (mm)	$U - U^*$	τ_E (s)	τ_E / τ_E^*
Interior point method	15.0062	0.0001	0.0283	0	2.55 (τ_E^*)	—
Active set method	15.0061 (D_C^*)	—	0.0283 (U^*)	—	3.25	1.3
Sequence quadratic optimization method	15.0061	0.0000	0.0283	0	5.94	2.3
AVG-based method: SQP pre-positioned	14.9869	-0.0192	0.0424	0.0141	0.21	0.08
AVG-based method: SQP pre-positioned and AVG re-positioned	14.9974	-0.0087	0.0294	0.0011	0.37	0.15
AVG-based method: SQP pre-positioned and AVG double re-positioned	15.0049	-0.0012	0.0283	0	0.47	0.18
AVG-based method: Centre pre-positioned	15.0060	-0.0001	0.0283	0	0.16	0.06

5.3 Algorithm improvement

In order to reduce the error caused by the positional changes, the points can be re-positioned, and the last position in the AVG based evaluation is an option. Then, the motion row vector A_i could be reconstructed according to the re-position, and a result of greater accuracy would be obtained. In this way, the algorithm is improved, programmed, and run to get the answer in MATLAB (R2014a), as shown in Table 1.

When re-position was performed once, the limit equivalent size was 14.9974 mm (0.0087 mm smaller than the standard value), the uncertainty of the result was 0.0294 mm (0.0011 mm greater than the standard uncertainty), and the total calculation time was 0.37 seconds (0.16 seconds more than the original AVG based method and 15% of the standard time). When re-position was performed twice, the size was 15.0049 mm (0.0012 mm smaller than the standard value), the uncertainty of the result was 0.0283 mm (as same as the standard uncertainty), and the total time was 0.47 seconds (0.26 seconds more than the once re-positioned method and 18% of the standard time). Thus, it is revealed that re-position (or a good original position) benefits the AVG based evaluation.

In addition, as the DMMR coaxiality tolerance was a boundary-based tolerance, we tried to pre-position the measurement obey the rule ‘minimum limits’ instead of the ‘LSQ’ method, where ‘minimum limits’ request that sum of the maximum and the minimum coordinates (of datum shaft) on each axis should be zero. This worked very well that the limit equivalent size was 15.0060 mm (0.0001 mm smaller than the standard value), the uncertainty of the result was 0.0283 mm (as same as the standard uncertainty), and the total calculation time was 0.16 seconds (0.05 seconds less than the original AVG based method and 6% of the standard time), as shown in [Table 1](#).

6 Conclusion

According to the geometric constraint of DMMR coaxiality tolerance and its real gauge based verification, the geometry and motion model of the adaptive virtual gauge (AVG) were proposed. Geometry and motion of the AVG and measurements were analyzed and modeled. Based on the model, a method for evaluating the DMMR coaxiality was proposed. The flow chart of the evaluation, the steps, and the necessary mathematical equations and conditions were given. The method to evaluate the uncertainty of the result was also provided.

Taking the evaluation of an actual stepped shaft as an example, the proposed method was implemented in MATLAB (R2014a). The calculation results show that the DMMR coaxiality of the actual part was not proved to meet the tolerance requirements. Compared with the classic evaluation methods based on optimization, the proposed method was 12 times faster, but the accuracy of the solution should be improved by using a more accurate original position of the measurements.

The proposed method was improved by performing one re-position and two re-positions. After two re-positions, the result of the improved method (15.0049 mm) was close enough (-0.0012 mm) to the classic value (15.0061 mm), while the uncertainty was the same as the classic uncertainty (0.0566 mm). Meanwhile the computation speed was 3.8 times faster than the fastest classic method.

The rule ‘minimum limits’ was also employed to pre-position the measurements, and showed high accuracy and speed. The result was 0.0001 mm smaller than the classic result, the uncertainty was as the same as the classic uncertainty, and the calculation time was only 6% of the classic time.

Measurements of the toleranced shaft and its datum shaft of DMMR coaxiality affect the evaluation result simultaneously and non-linearly. Thus, measurement uncertainties of the toleranced shaft and its datum shaft would not linearly add to the uncertainty of the result. We discussed the worst case to ensure a reliable uncertainty in this paper. As pre-position (or re-position) got better and better, and the expand uncertainty became smaller and smaller. However, when the actual datum shaft is not so big, the deviation of its size may not change the result. As a result, we evaluated and obtained a reliable expand uncertainty, but it is too larger to not accept some acceptable parts sometimes. Thus, we are going to investigate a reliable and more accurate method to evaluate the propose method, in the future.

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Appendix A. Discussion on motion state of contact points

In this appendix, the basic motion equation group is discussed to identify virtual constraints, virtual contact and over-constraints as followed.

Let the last contact point set be $\{p_i | i \in Q^{(k-1)} \subseteq I\}$, and let the current contact point set be $\{p_i | i \in Q^{(k)} \subseteq I\}$ with $Q (\leq 5)$ contact points and the new contact point p_q . In this scenario, $Q^{(k-1)} \subset Q^{(k)}$. By substituting $Q^{(k-1)}$ into Equation (23), the equation group for the $(k-1)$ -th contact can be obtained by Formula (23).

$$A^{(k-1)} \Psi^{(k-1)} = b^{(k-1)}, \quad (23)$$

where $A^{(k-1)} = [\dots, A_i^T, \dots]^T$ is the motion coefficient matrix; $b^{(k-1)} = [\dots, b_i, \dots]^T$ is the motion constant term of the contact point set, and $b_i^{(k-1)}$ is the b_i at the $(k-1)$ -th contact; $\Psi^{(k-1)}$ is the result at the $(k-1)$ -th contact; and $i \in Q^{(k-1)}$.

Since the AVG can push the measurement set through the contact set, Equation (23) has a solution. Thus, $\text{rank}(A^{(k-1)}) = \text{rank}(A^{(k-1)}|b^{(k-1)})$. Furthermore, as there are no virtual constraints, virtual contacts, or over-constraints, $\text{rank}(A^{(k-1)}) = \text{rank}(A^{(k-1)}|b^{(k-1)}) = Q-1 =$ the amount of contact points at the $(k-1)$ -th contact.

By substituting the current contact point set $Q^{(k)}$ into Equation (6), the equation group for the k -th contact can be obtained by Formula (24).

$$A^{(k)} \Psi^{(k)} = b^{(k)}, \quad (24)$$

where $A^{(k)} = [\dots, A_i^T, \dots]^T$ is the motion coefficient matrix at the k -th contact; $b^{(k)}$ is the motion constant term of the contact point set at the k -th contact, and $b_i^{(k)}$ is the b_i at the k -th contact; $\Psi^{(k)}$ is the result at the k -th contact; and $i \in Q^{(k)}$.

There are three states of the ranks of the motion coefficient matrix $A^{(k)}$ and the augmented matrix $(A^{(k)}|b^{(k)})$, and they are displayed and discussed bellow.

- (1) **State 1:** $\text{rank}(A^{(k)}) = \text{rank}(A^{(k)}|b^{(k)}) = Q-1 \leq 4$, when there is a virtual constraint (but this state will never happen).
- (2) **State 2:** $\text{rank}(A^{(k)}) = \text{rank}(A^{(k)}|b^{(k)}) = Q \leq 4$, when all contact points are physical contact points, and the motion state can be solved directly, as shown in Figure 2.
- (3) **State 3:** $Q-1 = \text{rank}(A^{(k)}) < Q = \text{rank}(A^{(k)}|b^{(k)}) \leq 5$, when there may be:
 - (a) A single virtual contact point, as shown in Figure 4(a).
 - (b) Several points, each of which could be a single virtual contact point. Then any one of them could be accepted as a single virtual contact point, as shown in Figure 4(b).
 - (c) Multiple points which could be virtual contact points simultaneously. Then there are at least one of them which could be accepted as a single virtual contact point, as shown in Figure 4(c).
 - (d) Over-constraints. Then the AVG has become its minimum, as shown in Figure 4(d).

(1) **State 1.** $\because \bar{A}_q^{(k)} = A^{(k-1)}$, where is the q reduced coefficient matrix, at the k -th touch.

$\therefore \text{rank}(\bar{A}_q^{(k)}) = \text{rank}(A^{(k-1)}) = \text{rank}(A^{(k)}|b^{(k)}) = Q-1 \leq 4$,

$\therefore \exists \beta = [\beta_1, \beta_2, \dots, \beta_{Q-1}]$:

$$\beta \left(\bar{A}_q^{(k)} \middle| \bar{b}_q^{(k)} \right) = \left(A_q \middle| b_q^{(k)} \right). \quad (25)$$

(i) If the SCD shrinks to the datum's MMVS at the k -th contact, the SCT should be established, and the last contact points are from the datum, and the new contact point is from the tolerated shaft.

$\therefore \bar{b}_q^{(k)}$ is a zero column vector θ with $Q-1$ elements, and one $b_q^{(k)} = b > 0$.

$\therefore \forall \beta: \beta \bar{b}_q^{(k)} = 0 \neq b = b_q^{(k)}$.

$\therefore \forall \beta: \beta(\bar{A}_q^{(k)} | \bar{b}_q^{(k)}) = (A_q | b_q^{(k)})$, which contradicts Equation (25).

(ii) If the SCD does not shrink to the datum's MMVS at the k -th contact, the measurement set keeps moving or remains static, and each uniform speed b_i will not change. Substituting $\bar{b}_q^{(k)} = b^{(k-1)}$ and $\bar{A}_q^{(k)} = A^{(k-1)}$ into Equation (25), then:

$$\exists \beta: \beta A^{(k-1)} = A_q, \text{ and } \beta b^{(k-1)} = b_q^{(k)} = b_q^{(k-1)}.$$

$\therefore$ Each side of Equation (23) is multiplied by β on the left and the result can be simplified as:

$$u_q^{(k-1)} = A_q \Psi^{(k-1)} = b_q^{(k-1)}.$$

This means that after the $(k-1)$ -th contact, the point p_q and its corresponding point P_i move at the same speed and will not contact. In short, **state 1** will not appear in any scenario.

(2) **State 2.** If all contact points are physical ones, Equation (24) has at least one solution, and any of them can be accepted as the rigid body's new motion state.

(3) **State 3.** In this state, over-constraint or virtual contact (Figure 9) occurs.

Figure 9. Virtual contacts: (a) a single virtual contact point, (b) multiple virtual contact points, or (c) combined virtual contact points.

(a) If there is a single virtual contact point p_i , as shown in Figure 9(a), then:

$$\exists \bar{\Psi}_i^{(k)}: \bar{A}_i^{(k)} \bar{\Psi}_i^{(k)} = \bar{b}_i^{(k)}, \text{ and } A_i \bar{\Psi}_i^{(k)} > b_i^{(k)}. \quad (26)$$

Also $\therefore \text{rank}(A^{(k)}) < \text{rank}(A^{(k)} | b^{(k)})$,

$$\therefore \forall \bar{\Psi}_i^{(k)} \in \left\{ \bar{\Psi}_i^{(k)} \mid \bar{A}_i^{(k)} \bar{\Psi}_i^{(k)} = \bar{b}_i^{(k)} \right\}: A_i \bar{\Psi}_i^{(k)} \neq b_i^{(k)}.$$

$$\therefore \forall \bar{\Psi}_i^{(k)}: \text{if } \bar{A}_i^{(k)} \bar{\Psi}_i^{(k)} = \bar{b}_i^{(k)}, \text{ then } A_i \bar{\Psi}_i^{(k)} > b_i^{(k)}.$$

That is, there will be a one-reduced equation group $\bar{A}_i^{(k)} \bar{\Psi}_i^{(k)} = \bar{b}_i^{(k)}$, and any of its solutions will satisfy Equation (26). Therefore, by checking each one-reduced equation group, the single virtual contact point would be found (if there is one). Then it would be removed from the contact point set.

(b) Obviously, if there are several contact points satisfying Equation (26), as shown in Figure 9(b), one of them can and shall be found and removed as a virtual contact point in the manner described in **state 3(a)**.

(c) If there are combined virtual contact points $\mathbf{p}_{i \in L \subset Q}$ and the amount of them is L , $1 < L < Q$, as shown in [Figure 9\(c\)](#), then:

$$\exists \bar{\Psi}_L^{(k)} : \bar{A}_L^{(k)} \bar{\Psi}_L^{(k)} = \bar{\mathbf{b}}_L^{(k)}, \text{ and } A_i \bar{\Psi}_L^{(k)} > b_i^{(k)} \Big|_{i \in L} .$$

$$\therefore \exists \bar{\Psi}_L^{(k)} : A^{(k)} \bar{\Psi}_L^{(k)} = \mathbf{b}^{(k)} + \sum_{i \in L} \Delta \mathbf{b}_i , \quad (27)$$

where $\Delta \mathbf{b}_i = [\mathbf{0}, \Delta b_i, \mathbf{0}]^T$ is a column vector with $Q-1$ zeros, $\Delta b_i > 0$, and the row of Δb_i is the same as the row of A_i in $A^{(k)}$.

Also $\therefore \text{rank}(A^{(k)}) = Q-1$.

$\therefore$ There is at most one $i = j \in L_j$ that makes $\text{rank}(\bar{A}_i^{(k)}) < Q-1$.

Let $L_{-j} \cup \{j\} = L$ and $j \notin L_{-j}$, then:

$\forall i \in L_{-j} : \text{rank}(\bar{A}_i^{(k)}) = Q-1$.

$\therefore \forall i \in L_{-j} : \exists \bar{\Psi}_i^{(k)} : \bar{A}_i^{(k)} \bar{\Psi}_i^{(k)} = \bar{\mathbf{b}}_i^{(k)} .$

$$\therefore \forall i \in L_{-j} : \exists \bar{\Psi}_i^{(k)} : A^{(k)} \bar{\Psi}_i^{(k)} = \mathbf{b}^{(k)} + \Delta \mathbf{b}_i^* , \quad (28)$$

where $\Delta \mathbf{b}_i^* = [\mathbf{0}, \Delta b_i^*, \mathbf{0}]^T$ is a column vector with $Q-1$ zeros, $\Delta b_i^* \neq 0$, and the row of Δb_i^* is the same as the row of Δb_i .

(i) If $\exists i \in L_{-j}$, and $\bar{\Psi}_i^{(k)} : A^{(k)} \bar{\Psi}_i^{(k)} = \mathbf{b}^{(k)} + \Delta \mathbf{b}_i^*$ and $\Delta b_i^* > 0$, then [Equation \(26\)](#) is satisfied. That is, the corresponding $\mathbf{p}_i$ is a single virtual contact point, and it must be found and removed in the manner described in **state 3(a)**.

(ii) If $\forall i \in L_{-j}$ and $\bar{\Psi}_i^{(k)} \in \left\{ \bar{\Psi}_i^{(k)} \mid A^{(k)} \bar{\Psi}_i^{(k)} = \mathbf{b}^{(k)} + \Delta \mathbf{b}_i^* \right\} : \Delta b_i^* < 0$, then:

A focus point $\mathbf{p}_F$ shall be selected. If there is a j that makes $\text{rank}(\bar{A}_j^{(k)}) < Q-1$, then $F = j$; otherwise, F could be any one in L .

Let $L_F \cup F = L$ and $F \notin L_F$, and substitute $i \in L_F$ into [Equations \(27\) and \(28\)](#), then:

$$\forall i \in L_F : \exists \bar{\Psi}_i^{(k)} \text{ and } k_i : k_i A^{(k)} \bar{\Psi}_i^{(k)} = k_i \mathbf{b}^{(k)} - \Delta \mathbf{b}_i , \quad (29)$$

where $k_i = -\Delta b_i / \Delta b_i^* > 0$.

Substitute all $i \in L_F$ and F into [Equations \(27\) and \(29\)](#) and sum them, then:

$$\exists \bar{\Psi}_L^{(k)} , \bar{\Psi}_i^{(k)} \text{ and } k_i : A^{(k)} \bar{\Psi}_F^{(k)} = \mathbf{b}^{(k)} + \Delta \mathbf{b}_F / K , \quad (30)$$

where

$$\bar{\Psi}_F^{(k)} = \left(\bar{\Psi}_L^{(k)} + \sum_{i \in L_F} k_i \bar{\Psi}_i^{(k)} \right) / K ;$$

$$K = 1 + \sum_{i \in L_F} k_i > 0 .$$

That is, $\exists i = F$ and $\bar{\Psi}_i^{(k)} = \bar{\Psi}_F^{(k)} : \bar{A}_i^{(k)} \bar{\Psi}_i^{(k)} = \bar{\mathbf{b}}_i^{(k)}$, and $A_i \bar{\Psi}_i^{(k)} > b_i^{(k)}$, which satisfies [Equations \(26\)](#) and indicates that $\mathbf{p}_F$ is a single virtual contact point. In addition, $\mathbf{p}_F$ is sure and necessary to be found and removed in the manner described in **state 3(a)**.

(d) If every one-reduced equation group is checked, and no virtual contact point is found, then over-constraint is formed. In

this state, the measurement set can only remain static, and the AVG can no longer shrink, as shown in Figure 2(d).

Nomenclature

<i>Variables</i>	Explain	<i>Variables</i>	Explain
A	Motion coefficient matrix	p_i	The i -th measurement point
A_i	Motion row vector of p_i	p_F	Focus point
$\bar{A}_S$	Reduced coefficient matrix according to S	p_q	The new contact point
b	Motion constant term	Q	Contact point set
b_i	Uniform speed of point P_i	Q, q	Amount and number of current contact points
$\bar{b}_S$	Reduced motion constant term according to S and S	R	Rotation matrix
β, β_i	A linear vector and its elements	R_C, R_D	Radius of the SCT and the SCD
D_A, D_b	Diameters of shafts A and b	R_{CD}	Radius of the SCT or the SCD
$\Delta b_i, \Delta b_i^*$	Residual vector for combined p_i and single p_i	R_{DM}	Radius of datum shaft's MMVB
$\Delta b_i, \Delta b_i^*$	Residual error for combined p_i and single p_i	r_i	Distance from p_i to the z -axis
dr_x, dr_y, dr_z	Rotation around the x, y and z axes	S	The numbers of removed contact points
dx, dy, dz	Translation along the x, y and z axes	S	The number of the only contact point that is removed
dx_i, dy_i, dz_i	v_i 's projection on the x, y and z axes	T	Translation vector
F	The number of Focus point	τ	During between two contacts
Ψ	Motion vector	τ_D	Expected time for SCD to shrink to MMVB
$\bar{\Psi}_S$	Reduced motion vector Ψ according to S	τ_i	Expected time for p_i to touch the AVG
I, i	Number set and the number (of ...)	u_D	Uncertainty of the final positions of datum shaft
j	Number of a special measurement point	u_{Dev}	Uncertainty caused by the deviation
k	Number of a current contact point	$u_{D,R}$	Rotation resultant uncertainty
L, L	The combined virtual contact points set and its amount	$u_{D,T}$	Translation resultant uncertainty
L_{-F}, L_{-j}	Resultant set by removing F or j from L	u_E	Uncertainty of measurement equipment
m, n	Numbers of datum and toleranced shafts' measurements	u_i	p_i 's normal linear velocity
N_i	Unit vector of the velocity direction of P_i	$u_{R,1}$	The maximum rotational deviation
P_i	Corresponding point of p_i	$u_{R,2}$	Allowed rotational deviation
$p_{C,ex,m}$	Measurements of the toleranced shaft	u_T	Uncertainty of the final positions of toleranced shaft
$p_{C,m}$	Pre-positioned coordinates of toleranced shaft	V_i, v_i	Velocities of P_i and p_i , respectively
$p_{D,ex,n}$	Measurements of the datum shaft	X_i, Y_i, Z_i	P_i 's projection on the x, y and the z axes
$p_{D,n}$	Pre-positioned coordinates of datum shaft	x_i, y_i, z_i	p_i 's projection on the x, y and the z axes

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