Sinking Bubbles in Stout Beers

W. T. Lee*

Department of Mathematics, University of Portsmouth,

Lion Gate Building, Lion Terrace, Portsmouth, UK.

S. Kaar and S. B. G. O'Brien

MACSI, Department of Mathematics and Statistics,

University of Limerick, Limerick, Ireland

(Dated: November 23, 2017)

Abstract

A surprising phenomenon witnessed by many is the sinking bubbles seen in a settling pint of stout beer. Bubbles are less dense than the surrounding fluid so how does this happen? Previous work has shown that the explanation lies in a circulation of fluid promoted by the tilted sides of the glass. However, this work has relied heavily on computational fluid dynamics (CFD) simulations. Here we show that the phenomenon of sinking bubbles can be predicted using a simple analytic model. To make the model analytically tractable we work in the limit of small bubbles and consider a simplified geometry. The model confirms both the existence of sinking bubbles and the previously proposed mechanism.

I. INTRODUCTION

One of the most important ways in which stout beers such as Guinness differ from other beers is that the mixture of dissolved gases within the beer includes nitrogen as well as carbon dioxide.[1] In most beers the only dissolved gas is carbon dioxide. The introduction of dissolved nitrogen into the gas mixture used to make the beer foam radically changes the appearance and taste of the beer, as well as affecting the way in which the beer must be poured or canned.[2] Nitrogen is less acidic in solution than carbon dioxide, giving stout beers a smoother, less acidic taste. Also, nitrogen is much less soluble than carbon dioxide so that, even though overall the dissolved gases in stout beers are at a higher pressure than in carbonated beers, the molar amount of the dissolved gases is actually much smaller. The low solubility of nitrogen is the reason why the head of a stout beer is much longer lasting than the head of a carbonated beer.[3] It also causes difficulties in making the beers foam which is why, unlike carbonated beers, stout beers require special technology in the tap or can: restrictor plates and widgets respectively.[4] The small amount of dissolved gases results in smaller bubbles in stout beers: stout beer bubbles are typically a tenth of a millimetre in size whereas in carbonated beers typical sizes are of the order of millimetres.[6]

The small bubbles of stout beers are behind many of the distinctive features of these beers. Small bubbles in the head are the reason for the creamy mouthfeel of stout beers and also play a role in the famous phenomenon of sinking bubbles.[7] However, as sinking bubbles have also been observed in other systems with larger bubbles (e.g. bubbles produced by a fizzing tablet in water[8]) the role the small bubbles plays may be simply to make the sinking bubbles easier to observe. (For the impatient drinker the small bubbles are also responsible for the long wait for a pint of stout beer to settle.) The origin of the sinking bubbles has long been controversial as indeed has been whether this happens at all or if the phenomena is an optical (or alcohol induced) illusion. The latter point was laid to rest by researchers who successfully videoed the sinking bubbles, showing that the phenomenon was due to a circulation within the glass with downwards currents close to the wall of the glass and opening the phenomenon up to scrutiny outside the pub.[8] The origin was also investigated via a series of computational fluid dynamics studies[9,10] which also found a circulatory flow within the glass resulting in the bubbles sinking due to the flow rather than rising due to their buoyancy. That is to say that although the bubbles are rising relative to the liquid due to their buoyancy, they are still falling relative to glass because the circulating liquid is falling faster than the bubbles are rising relative to the liquid. Finally, the origin of the circulatory flow was demonstrated as an example of the Boycott effect[11,12] promoted by the shape of the Guinness glass,[13] a factor which had not been fully investigated in previous studies of settling in stout beers.

A particularly persuasive argument of Ref. 13 was the experimentally confirmed prediction that both rising and sinking bubbles should be seen in a stout beer settling in a tilted measuring cylinder. However, one weakness in the argument was that it jumped from conceptual models straight to computational fluid mechanics models. An analytically tractable mathematical model capturing the essence of the phenomena would be valuable both to increase confidence that the explanation is correct and to build intuition regarding the phenomena. Here we report such a model taking inspiration from the tilted measuring cylinder experiment, which allows a number of simplifications to be made.

The structure of the remaining parts of the paper is as follows. In Sec. II we present a mathematical model of the motion of beer and bubbles in an idealised version of the tilted measuring cylinder geometry. The model is much simpler than the full set of equations describing bubbly flows typically solved by CFD simulations. We show that the slender nature of the geometry and small size of the bubbles allow us to justify these assumptions, which result in a set of decoupled equations in which we can independently solve for flows across the cylinder and along the cylinder. (Note that to keep the equations as simple as possible we will sometimes have to assume that bubbles are smaller than they are in reality.) A mathematical appendix discusses these assumptions in more detail. Sec. III discussed flow across the cylinder. In this direction bubbles and beer are constrained to flow in opposite directions leading to a slow flow in which a bubble free region forms on the lower edge of the cylinder and a bubble rich region forms at the upper surface of the cylinder. In Sec. IV we discuss the implications of the bubble free region along the lower edge of the cylinder for flow parallel to the axis of the cylinder—in this direction bubbles and beer are constrained to flow together. We show that sinking bubbles are predicted by this flow. Sec. V discusses how this model and the assumptions used to describe it relate to reality. Finally, conclusions are given in Sec. VI.

II. MATHEMATICAL MODEL

The flow of bubbles and beer in a 'tulip' pint glass is very complex, and can only really be addressed by computational fluid dynamics simulations. These simulations solve six partial differential equations (assuming the simulations take advantage of the cylindrical symmetry of the pint glass). Two equations describe the conservation of volume occupied by the beer and bubbles respectively. The remaining four equations are momentum equations: describing conservation of momentum of bubbles and beer in the z and r directions.

Modern computing hardware and algorithms can solve this complicated set of equations very rapidly. The simulations reported in Ref. 13 were run on a desktop computer. However, the ability to reproduce a phenomenon in a simulation does not always lead to a better understanding of that phenomenon, any more than observing it in the real world does. This can clearly be seen from the fact that it was 13 years after the first reported CFD simulation 'explaining' the sinking bubbles that the crucial role of the geometry of the glass in determining whether bubbles are seen to sink or rise was recognised.

In this paper we take inspiration from the measuring cylinder experiments discussed above and create a simplified set of equations which can describe this situation. Our approach is to derive a set of equations containing only those terms which physical intuition suggests are the most important and then use dimensionless numbers to confirm that the terms neglected are negligible. The geometric and physical parameters used are given in Table I.

The geometry under consideration is shown in Fig. 1. We consider a 'two-dimensional cylinder' consisting of two parallel plates tilted at an angle θ to the vertical. For simplicity the word 'cylinder' will still be used to describe the system. The height H is much greater than its length L. We take a coordinate system embedded in the cylinder so that the x-axis is perpendicular the axis of the cylinder and the y-axis is parallel to the axis of the cylinder. The variables of the system are

- ϕ the volume fraction of the bubbles
- u the velocity of bubbles in the x direction
- U the velocity of beer is the x direction
- v the velocity of bubbles in the y direction

TABLE I. Physical and geometric properties.

Parameter	Value	Reference
ρ_{beer}	$1007\,\mathrm{kg\,m^{-3}}$	7
ρ_{bubble}	$1.223\,\mathrm{kg\,m^{-3}}$	13
μ	$2.06\times10^{-3}\,\mathrm{Pa\,s}$	7
r	$61\,\mathrm{\mu m}$	6
θ	5°	
g	$9.81\,\mathrm{m\,s^{-2}}$	
L	$2\,\mathrm{cm}$	
ϕ_0	0.02	13
ϕ_{Head}	0.80	
u_{Stokes}	$3.45\times10^{-4}\,\mathrm{m\,s^{-1}}$	
v_{Stokes}	$3.95\times10^{-3}\,\mathrm{m\,s^{-1}}$	

- V the velocity of beer in the y direction
- p the pressure in the system

In discussions below the words ‘horizontal’ and ‘vertical’ and ‘up’ and ‘down’ refer to the x-y coordinate system embedded in the cylinder.

The key assumptions we make are that the bubble size is small (the question of what this means in practice is discussed below) and that $L \ll H$. In most cases the actual size of the bubbles $r = 61\,\mathrm{\mu m}$, will be sufficiently small to justify the simplifications we make: we will discuss in more detail cases in which this is not true. The fact that $L \ll H$ suggests that there will be a slow variation in properties in the y direction compared to the x direction. Thus we assume that all the system variables are independent of y. That is to say that ϕ, u, v, U and V are functions of x and t only. (The pressure, p, is a special case that will be discussed later.) Note that it does *not* follow from this assumption that $v = V = 0$: although quantities do not, to a first approximation, depend on y there is no prohibition against vectors pointing in the y direction.

When coupled with the assumption that both the bubbles and the beer are incompressible

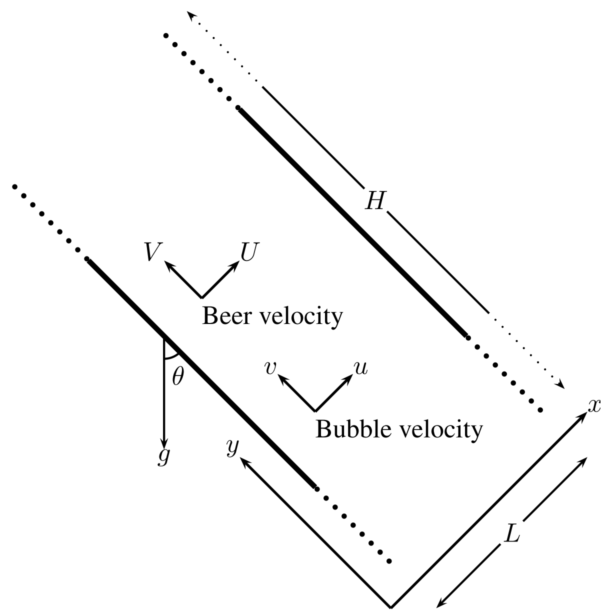

FIG. 1. Geometry of the tilted cylinder showing the coordinate system embedded in the cylinder and the components of the velocity fields of bubbles and beer. Note that the tilt has been exaggerated in this diagram.

(also assumed by CFD simulations) this has important implications for the types of flow that are possible in the x and y directions. No net flow is possible in either direction. However flow through a horizontal surface need not be uniform, so a circulatory flow in which the flow is downwards at some locations and upwards in other locations is possible. In contrast flow through any vertical surface must be zero. Stating these conditions as equations we have

$$0 = \phi u + (1 - \phi)\, U, \tag{1}$$

$$0 = \int_0^L \left[\phi v + (1 - \phi)\, V\right]\, \mathrm{d}x. \tag{2}$$

In particular these equations tell us that for motion in the x direction beer and bubbles must be travelling in opposite directions whilst for flow in the y direction there is no prohibition against beer and bubbles moving in the same direction—so as long as the overall flow is circulatory in nature so there is no net flow through a horizontal surface.

The small size of bubbles leads to a number of further simplifications. The trajectories of small bubbles are dominated by drag forces. Thus, where net flows are possible, i.e., in the y direction, we expect any difference between the velocities of the bubbles and beer to be negligible compared with the overall velocities. So for flows in the y direction it makes sense to assume $v = V$ and model the flow of the beer and bubbles together as 'bubbly beer' with a single velocity $\bar{V}$ but with a non-uniform density $\rho = (1-\phi)\,\rho_{\mathrm{beer}} + \phi\rho_{\mathrm{bubbles}}$. Additionally we can use $\rho_{\mathrm{bubbles}} \ll \rho_{\mathrm{beer}}$ to justify the approximation $\rho \approx (1-\phi)\,\rho_{\mathrm{beer}}$.

We cannot make this assumption for flow in the x direction. This is an advantage however, since it suggests a separation of timescales. Since bubbles and beer are constrained to move in opposite directions in the x direction this means that the timescale associated with flow in the x direction will be much longer than the timescale associated with flow in the y direction. This means that flow in the y direction can be considered as quasi-static and we can neglect time derivatives for flow in the y direction.

These considerations suggest the following equations for $\phi(x,t)$, $u(x,t)$, $U(x,t)$ and $\bar{V}(x,t)$. For flow in the x direction:

$$\frac{\partial \phi}{\partial t} = -\frac{\partial\,(u\phi)}{\partial x}, \tag{3}$$

with a constitutive equation describing $u - U$ as a function of ϕ described in Sec. III. For flow in the y direction:

$$\bar{V} = \phi v + (1-\phi)\,V \tag{4}$$

$$0 = \int_0^L \bar{V}\,\mathrm{d}x \tag{5}$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos(\theta) + \mu \frac{\partial^2 \bar{V}}{\partial x^2} \tag{6}$$

where the pressure p is discussed in more detail in Sec. IV. In the sections below we show that this system of equation is sufficient to produce a model in which sinking bubbles appear.

III. FLOW ACROSS THE CYLINDER

As discussed above, the equations describing flow across the cylinder are

$$0 = \phi u + (1-\phi)\,U, \tag{7}$$

$$\frac{\partial \phi}{\partial t} = -\frac{\partial\,(u\phi)}{\partial x}. \tag{8}$$

The first equation follows from the incompressibility of beer and bubbles, the second describes conservation of bubbles.

As currently stated the system is underdetermined since we have two equations and three fields to solve for: ϕ, u and U. In a computational fluid dynamics simulation these equations would be closed by the inclusion of momentum equations. However, here we follow Kynch[14] in closing the system of equations by assuming that the relative velocity of the bubbles and beer only depends on ϕ:

$$u - U = u_{\text{Stokes}} f(\phi)\,, \tag{9}$$

where u_{Stokes} is the (horizontal) Stokes velocity,

$$u_{\text{Stokes}} = \frac{2}{9}\,\frac{r^2\left(\rho_{\text{beer}} - \rho_{\text{bubbles}}\right) g \sin\theta}{\mu}. \tag{10}$$

To solve these equations we eliminate u and U to get a partial differential equation for ϕ.

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x}\left[u_{\text{Stokes}} \phi \left(1 - \phi\right) f(\phi)\right] = 0 \tag{11}$$

This equation can be solved using initial condition $\phi(x, 0) = \phi_0 \approx 0.02$, and boundary conditions $\phi(0, t) = 0$, $\phi(H, t) = \phi_{\text{Head}}$.

A variety of forms can be taken for $f(\phi)$,[15] here for simplicity we assume that bubbles either move at the Stokes velocity when the beer density is low or come to rest when the bubble density is at a similar level to that found in the foam forming head of a pint:

$$\left(1 - \phi\right) f(\phi) = 1, \qquad \phi < \phi_{\text{Head}}, \tag{12}$$

$$\left(1 - \phi\right) f(\phi) = 0, \qquad \phi \geq \phi_{\text{Head}}, \tag{13}$$

where $\phi_{\text{Head}} \approx 0.8$ is the bubble volume fraction of the head of a pint of beer.

Since this equation is a hyperbolic first order partial differential equation it can be solved using the method of characteristics. This shows that the system separates into three regions. A region containing only beer ($\phi = 0$), a region containing bubbly beer ($\phi = \phi_0$) and a region containing foam ($\phi = \phi_{\text{Head}}$). There is a discontinuous change in ϕ at the interfaces between these regions, so to find the positions of these interfaces as a function of time the Rankine-Hugoniot jump conditions for describing shocks must be used to find the location of the shock separating beer from bubbly beer $x_1(t)$, and the location of the shock separating bubbly beer from foam $x_2(t)$.

However, once the structure of the solutions has been recognised it is much easier to deduce the locations of the shocks from physical principles. The interface between beer and

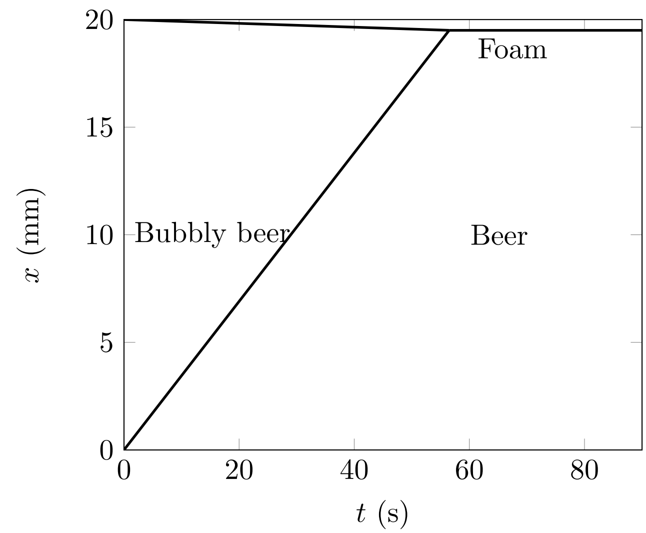

FIG. 2. In the x direction the system partitions into regions containing beer, bubbly beer and foam.

bubbly beer, x_1, must be moving upwards from $x = 0$ at the same speed as the bubbles so

$$x_1(t) = u_{\text{Stokes}} t.$$

The location of the second shock x_2 separating bubbly beer and foam can be calculated from conservation of bubbles. That is to say we must have $(L - x_2)\,\phi_{\text{Head}} + (x_2 - x_1)\,\phi_0 = L\phi_0$. We can solve this equation to give

$$x_2(t) = L - \frac{\phi_0 u_{\text{Stokes}} t}{(\phi_{\text{Head}} - \phi_0)}$$

These results are illustrated in Fig. 2. Eventually these two shocks will collide to give a single interface separating beer from foam. However we will be most interested in what happens before then.

IV. FLOW ALONG THE CYLINDER

As discussed above flow parallel to the walls can be described in terms of the motion of a single fluid with a velocity $\bar{V}$ and density depending on x. The equations describing the flow are

$$0 = \int_0^L \bar{V}\,\mathrm{d}x, \tag{14}$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \cos(\theta) + \mu \frac{\partial^2 \bar{V}}{\partial x^2}. \tag{15}$$

The first equation follows from the fact that the bubbles and beer are incompressible, and that the end of the cylinder is closed. It states that there is no net flow through any horizontal surface. The second equation describes momentum transfer. Three aspects of this equation require further discussion.

The first consideration is that this equation assumes the fluid is Newtonian with the same viscosity as pure beer (Boussinesq approximation[5]). This is a reasonable assumption in the beer and bubbly beer regions, but foams typically have a non-Newtonian rheology in which a non-zero shear stress is needed to initiate flow cf. the behaviour of paints. For simplicity, here we assume that the imposed shear stress does not exceed this threshold: so we assume that the foam is motionless. Furthermore since $\phi_0 \ll \phi_{\text{head}}$ it follows that $L - x_2 \ll L$, so we approximate x_2 by L. We therefore assume that

$$\phi = \begin{cases} 0 & 0 \leq x < x_1 = u_{\text{Stokes}} t, \\ \phi_0 & x_1 \leq x \leq L, \end{cases} \tag{16}$$

and that the boundary conditions are $\bar{V} = 0$ when $x = 0$ or $x = L$.

The second consideration is the neglect of the inertia terms in the equation. This assumption is discussed in Appendix A 2. As discussed in that appendix this assumption is valid in the small bubble limit, but, strictly speaking, the size of bubbles actually found in stout beers are not small enough to justify this assumption. For simplicity we continue to make this assumption and discuss the consequences of relaxing it in Sec. V.

The third consideration is the role of the pressure p. Above it was stated that all the fields of the system ϕ, u, v, U, V were only dependent on x and independent of y. This is not quite true for p, here it is the pressure gradient $\partial_y p$ that is independent of y. In fact in order to preserve the y-independence of the velocities $\partial_y p$ must be independent of x too. Thus the y-component of the pressure gradient is a constant which we denote by p_y. The easiest physical picture of the role of this constant is as a Lagrange multiplier that enables us to impose the condition of no net flow through a horizontal surface.

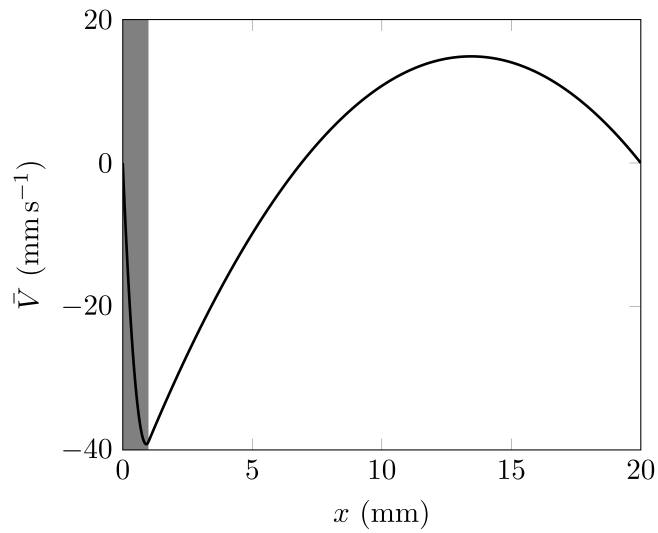

FIG. 3. Velocity in the y direction as a function of position in the x direction for the case in which $x_1 = L/20$. The shaded region shows the layer of pure beer. As can be seen the velocity is negative in the bubbly beer region, i.e. sinking bubbles are predicted.

Now that we know the pressure gradient is a constant we can solve Eq. (15) by integrating twice and choosing the constants of integration to impose the no-slip conditions at $x = 0$ and $x = L$. (This process imposes continuity of $\bar{V}$ and $\partial_x \bar{V}$ at $x = x_1$). The value of the constant p_y is chosen so that the $\bar{V}$ will satisfy Eq. (14). This gives

$$\bar{V} = -\frac{g\phi_0\rho_{\mathrm{beer}}\cos\theta}{2\mu L^3} x\,(L - x_1)^2\,(2x_1 L - xL - 2xx_1) \tag{17}$$

when $0 \le x < x_1$, and

$$\bar{V} = -\frac{g\,\phi_0\,\rho_{\mathrm{beer}}\,\cos\theta}{2\mu L^3} x_1^2\,(L - x)\left(L^2 + 2xx_1 - 3xL\right) \tag{18}$$

when $x_1 \le x \le L$.

Figure 3 shows a plot of $\bar{V}$ when $x_1 = 0.05L$. As the figure shows $\bar{V}$ is negative for $x \gtrsim x_1$ and thus this model correctly predicts sinking bubbles at the lower edge of the cylinder. (It also correctly predicts rising bubbles near the upper edge of the cylinder.)

V. DISCUSSION

As has been shown above the simple model presented above reproduces the phenomenon of sinking bubbles in stout beers. This model is an important confirmation of the arguments presented in Ref 13, since in that work the arguments were supported by computational fluid dynamics simulations. Computational fluid dynamics simulations are very general and contain all sorts of additional physical effects. Thus it is impossible to completely rule out other potential mechanisms behind the sinking bubbles. Unlike those simulations, the model presented here contains only the physical ingredients essential to the argument and it can be seen that sinking bubbles still emerge.

The model presented is applied to two-dimensional version of the experimentally observed measuring cylinder case. Thus the assumptions made in setting up the model are not directly applicable to the sinking bubbles seen in a tulip pint glass. Nevertheless the qualitative features of the phenomena are the same in each case. Below we discuss some of the other differences between the model presented and the real world phenomena.

One important assumption made (which is also commonly made in computational fluid dynamics simulations) is that the bubbles are monodisperse, i.e. all the same size. In reality there is a range of bubble sizes. Differently sized bubbles will rise at different rates and so in the real polydisperse case the sharp interfaces between regions of beer, bubbly beer and foam predicted in Sec. III will be replaced by transition regions in which ϕ gradually changes. However, this gradual rather than abrupt change will not affect the main conclusion that sinking bubbles will be observed.

The assumption that our variables are not functions of y is valid only far from the bottom and the top of the cylinder. Much more complex two dimensional flow patterns will be seen in these regions. It seems unlikely that these can be modelled without resorting to numerical simulations. However the existence of the bottom of the cylinder is important in our calculation since the impermeable base is the origin of the constraint that the net flow through a horizontal surface must be zero.

An additional assumption made was the neglect of inertia terms in the momentum equation for flow parallel to the walls of the cylinder. As noted in Sec. IV and Appendix A 2, whilst this assumption is valid in the limit of small bubbles, the bubbles found in stout beers are not small enough to justify this assumption. Employing this assumption removed any time derivative terms from the equation. Had this term been left in the velocity profile along the cylinder would have retained a memory of previous conditions. Thus whilst the quantitative details of the flow would change the qualitative aspects of the flow would have remained the same, in particular the phenomenon of sinking bubbles would still have be observed. A numerical calculation demonstrating this is discussed in Appendix B.

Finally the observed flow patterns of sinking bubbles are much more complex than has been described by this model: as is well known the sinking bubbles form waves. A one dimensional model of this phenomenon has been presented[6]. However the shear flow shown in Fig. 3 suggests an alternative mechanism based on shear instability. The most commonly discussed form of shear instability is the Kelvin-Helmholtz instability seen when there is a transverse discontinuity in the velocity. This would be observed in our model in the limit $\mu \to 0$. However shear instabilities are also possible in viscous fluids. In case of inviscid flows it is known that a strong indicator that a flow will be unstable is the existence of an inflection point at which the shear gradient $\partial_y^2 \bar{V}$ changes sign. Differentiation shows that $\partial_y^2 \bar{V}$ always changes sign at x_1, suggesting that such an instability is present.

$$\partial_y^2 \bar{V} = \frac{g\phi_0\rho_{\text{beer}}}{\mu\, L^3} \left(L - x_1\right)^2 \left(2x_1 + L\right) > 0 \qquad x < x_1 \tag{19}$$

$$\partial_y^2 \bar{V} = -\frac{g\phi_0\rho_{\text{beer}}}{\mu\, L^3} x_1^2 \left(3L - 2x_1\right) < 0 \qquad x > x_1 \tag{20}$$

A complete analysis of the instability of the flow is possible but would be very complex. Investigating the instability would involve a more complex series of equations with the missing t and x derivatives reinstated.

VI. CONCLUSIONS

The sinking bubbles of stout beers are an everyday example of a complex two phase flow phenomenon. We have shown that a relatively simple, analytically solvable mathematical model can explain this phenomenon. The model works in the limit of small bubble size and a long thin geometry.

ACKNOWLEDGMENTS

The authors gratefully acknowledge the support of the Mathematics Applications Consortium for Science and Industry (www.macsi.ul.ie) funded by the Science Foundation Ireland (SFI) Investigator Award 12/IA/1683.

Appendix A: Mathematical Appendix

In this section we give more details of the considerations used to develop the simplified system of equations used. The essence of our procedure is to use physical intuition develop equations which include only the most important terms. Having done this we confirm by calculating dimensionless numbers that the terms neglected will be small.

1. Horizontal Motion

Consider first the flow in the horizontal direction. Here we assumed that the momentum equation is dominated by a balance between the Stokes drag force and the hydrostatic pressure. This leads to the assumption that the velocity of the bubbles will be the Stokes velocity

$$u_{\text{Stokes}} = \frac{2}{9} \frac{r^2 \left(\rho_{\text{beer}} - \rho_{\text{bubbles}}\right) g \sin\theta}{\mu}. \tag{A1}$$

In making this assumption we are neglecting virtual mass forces. The magnitude of virtual mass forces acting on a single bubble will be

$$f_{\text{VM}} \sim \frac{C_{\text{VM}} \rho_{\text{beer}} u_{\text{scale}} r^3}{t_{\text{scale}}}, \tag{A2}$$

where C_{VM} is a dimensionless order-1 coefficient we take to be unity for simplicity here, u_{scale} and t_{scale} are characteristic velocity and time scales of the system. A sensible choice for the velocity scale would be the Stokes velocity u_{Stokes}, while a sensible choice for the time scale would be the the time it takes a bubble travelling at the Stokes velocity to traverse the system $t_{\text{scale}} = L/u_{\text{Stokes}}$

$$f_{\text{VM}} = \frac{\rho_{\text{beer}} u_{\text{Stokes}}^2 r^3}{L}. \tag{A3}$$

We can demonstrate that it is reasonable to neglect virtual mass forces in our equations by calculating a dimensionless number comparing the magnitude of virtual mass to the magnitude of drag forces (given by the Stokes drag law)

$$f_{\text{D}} = 6\pi\mu r u_{\text{Stokes}} \tag{A4}$$

$$\frac{f_{\text{VM}}}{f_{\text{D}}} = \frac{\rho_{\text{beer}} u_{\text{Stokes}} r^2}{6\pi\mu L} \approx 1.6 \times 10^{-6} \ll 1 \tag{A5}$$

This demonstrates that virtual mass forces are negligible compared to drag forces, and can safely be neglected in the equations.

2. Vertical Motion

The equations describing the vertical velocity field rely on two assumptions. These are that (1) bubble motion relative to beer motion can be neglected so that flow in the vertical direction can be modelled as that of a single fluid; (2) the velocity of the fluid is determined by a balance between weight/buoyancy forces and viscous forces, with inertial forces neglected.

Weight and buoyancy forces can be estimated as $f_{\text{buoyancy}} = \rho_{\text{beer}}\phi_0 g\cos(\theta)$, while viscous forces can be estimated as $f_{\text{viscous}} = \mu V_{\text{scale}}/x_{\text{scale}}^2$. Taking x_{scale} to be L, the extent of the system allows us to estimate V_{scale} as

$$V_{\text{scale}} = \frac{\rho_{\text{beer}}\phi_0 g\cos(\theta)\,L^2}{\mu} \tag{A6}$$

by balancing viscous and buoyant forces.

The validity of assumption (1) that bubbles and beer can be considered as moving together can be investigated by comparing the magnitude of V_{scale} with the velocity of the bubbles relative to the beer, approximated by the vertical Stokes velocity. (Note that horizontal and vertical Stokes velocities are different.)

$$\frac{v_{\text{Stokes}}}{V_{\text{scale}}} = \frac{\mu v_{\text{Stokes}}}{g\phi_0\rho_{\text{beer}}L^2\cos\theta} \approx 1.0\times 10^{-4} \ll 1. \tag{A7}$$

Since the relative velocity is much smaller that the overall velocity it makes sense to consider the bubbles and beer as moving together and describe their motion by a single combined equation.

The final assumption is the neglect of inertial forces. The magnitude of these can be approximated by $f_{\text{inertial}} = \rho_{\text{beer}}V_{\text{scale}}/t_{\text{scale}}$, where the relevant timescale t_{scale} is that of the motion of bubbles in the horizontal direction since it is the horizontal motion of bubbles driving the whole process. For our analysis to be correct the ratio of inertial to viscous forces should be small. In fact we have

$$\frac{f_{\text{inertial}}}{f_{\text{viscous}}} \approx 3, \tag{A8}$$

This shows our analysis is not strictly correct. However since the ratio is proportional to r^2 (via the Stokes velocity), so if the bubble radius is small enough the analysis will be valid. A numerical calculation of the velocity field with inertial terms included is discussed in Appendix B.

Appendix B: Numerical treatment of Inertia Terms.

As discussed above the assumption that flow in the y direction could be treated as quasistatic made in the main body of the paper is valid in the limit of small bubble sizes but only for bubble sizes significantly smaller than are observed in practice. If we relax this assumption the equations that must be solved are

$$\rho_{\mathrm{beer}} \frac{\partial \bar{V}}{\partial t} = -p_y - \rho_{\mathrm{beer}} g \left(1 - \phi\right) + \mu \frac{\partial^2 \bar{V}}{\partial x^2} \tag{B1}$$

as before we are make a Boussinesq assumption[5] in taking the density of fluid to be the density of bubble free beer. The bubble volume fraction is given by

$$\phi = \begin{cases} 0 & x < u_{\mathrm{Stokes}} t \\ \phi_0 & x \geq u_{\mathrm{Stokes}} t \end{cases} \tag{B2}$$

and p_y is chosen to impose

$$0 = \int_0^L \bar{V} \, \mathrm{d}x. \tag{B3}$$

This can be discretised with implicit Euler timestepping as

$$\rho_{\mathrm{beer}} \frac{v_i^{\alpha+1} - v_i^{\alpha}}{\delta t} = -p_y - \rho_{\mathrm{beer}} g \left(1 - \phi_i^{\alpha}\right) + \mu \frac{v_{i+1}^{\alpha+1} - 2 v_i^{\alpha+1} - v_{i-1}^{\alpha+1}}{\delta x^2} \tag{B4}$$

where v_i^{α} is the velocity at coordinate $i\delta x$ and time $\alpha \delta t$. In matrix form this can be written as

$$\mathbf{M} \mathbf{v}^{\alpha+1} = -p_y \mathbf{1} + \mathbf{b} \tag{B5}$$

where $\mathbf{M}$ is a tridiagonal matrix, $\mathbf{1}$ is a vector of 1's and $\mathbf{b}$ is a vector. p_y is chosen so that $\mathbf{v} \cdot \mathbf{1} = 0$ (the discrete equivalent of Eq. (14))

$$p_y = \frac{\mathbf{1}^{\mathrm{T}} \mathbf{M}^{-1} \mathbf{b}}{\mathbf{1}^{\mathrm{T}} \mathbf{M}^{-1} \mathbf{1}}. \tag{B6}$$

The results of a numerical calculation of the velocity is shown in Fig. 4 and shows that sinking bubbles are still expected.

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* Currently at: School of Computing and Engineering, University of Huddersfield, Queensgate, Huddersfield, UK.; w.lee@hud.ac.uk; www.industrial-maths.com

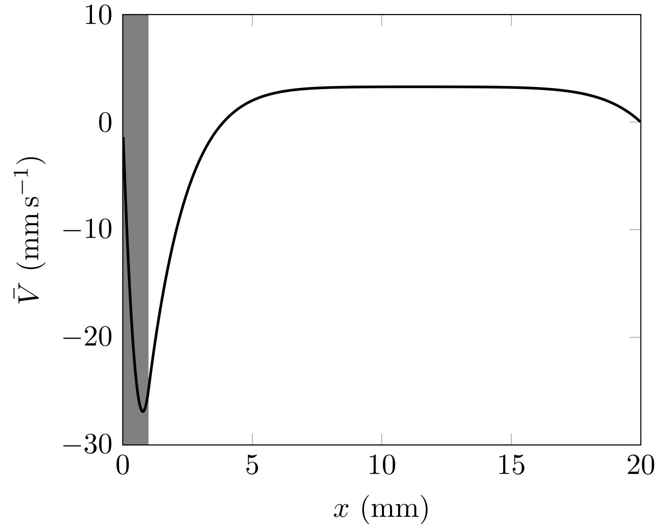

FIG. 4. Numerical calculation of velocity in the y direction as a function of position in the x direction. The bubble free region is shaded in grey.

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