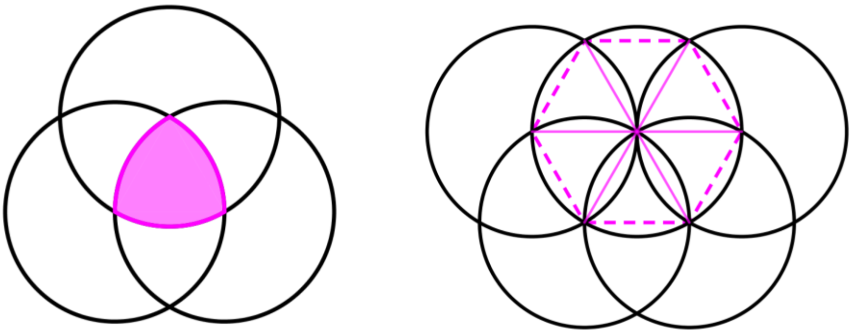
The Compass and Straight Edge – Alternative Approaches to teaching constructions

Constructions get a bad reputation in schools in England. They've been pushed down into the Key Stage 3 Program of Study, and are often met with quiet disdain by those who teach the topic. I know this because not only have I been that teacher, but I speak to many such teachers in my job role as a teacher trainer, and when I led a workshop about constructions last year, the story I heard from several participants was very familiar – they've always hated it. In fairness, there are indeed a multitude of barriers to learning when teaching this topic, and each must be carefully considered and planned for in order to overcome them all.

The first issue is one of equipment – in no other mathematical topic do we need to pay such careful attention to the availability and suitability of tools in the classroom. Relying on students themselves to bring a pair of compasses, ruler, pencil and pencil sharpener is likely to end with a shortage of several, if not all of these tools. Similarly, the expectation that your own equipment drawers will contain enough of each, is perhaps a little too optimistic. An experienced teacher recalls all too vividly the lesson that fell apart due to pupils sharing a ruler between three, or the queue of twelve students at the bin waiting to use the coveted sharpener before they can begin their quest to construct a perpendicular bisector. As such it is recommended that prior to teaching constructions, departmental resources are pooled and checked to avoid ruler rations and compass curfews. On the subject of compasses, it should be further noted that school compasses are likely to be loose, so a set of complementary screwdrivers is generally recommended. Another common mistake in teaching constructions is when we make the assumption that students can use, or will be able to use, a pair of compasses with almost no prior practice or instruction. It's unlikely that diving straight into angle bisectors will yield strong results without first practicing drawing a perfect circle. This needs not be a tedious task and could be viewed as an opportunity for some creativity. Consider the constructions below using concentric circles and straight lines.

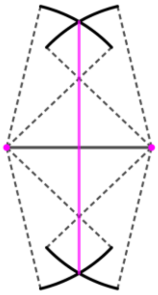


The constructions serve the dual purpose of practicing the art of accurate drawing using a pair of compasses and a straight edge, but also enabling the opportunity to notice the *reuleaux triangle* (left) and the circumscribed hexagon (right) whose perimeter is equal to six radii. Notice also that its perimeter is just a little shy of the circumference of the circle (it fits tightly inside it).

In other words, three diameters is a little less than the circumference, which in turn can lead to a nice discussion about pi.

The perpendicular bisector

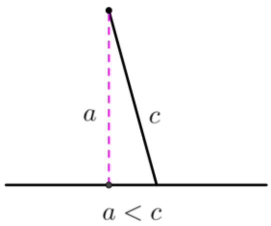
The classic approach to constructing a perpendicular bisector (from the Latin *perpendiculum*, meaning plumb line) is as follows.



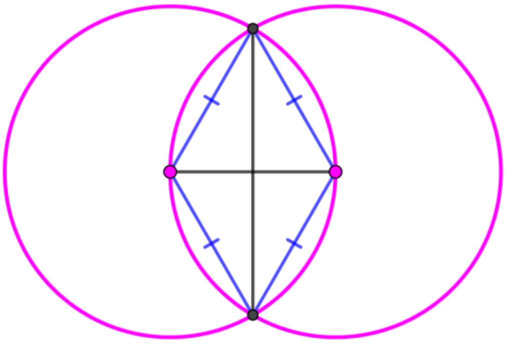
In terms of communicating the procedure, it is good practice to both demonstrate constructing the diagram live, using some kind of visualizer, and also presenting a number of animated gifs as reference when students inevitably forget what you did and in what order.

So, in terms of conceptual understanding, what is actually happening and why does it work?

The central premise is that the shortest distance between a point and a line is a perpendicular line. You can convince yourself that this is true by drawing any non-perpendicular line (c), then constructing a right triangle *using that line (a)*. By Pythagoras the original line is not the shortest:

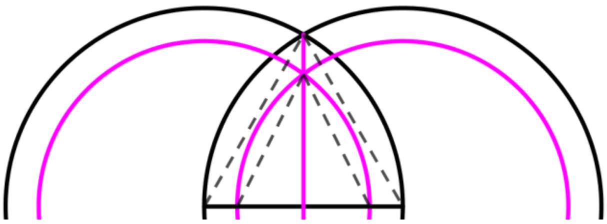


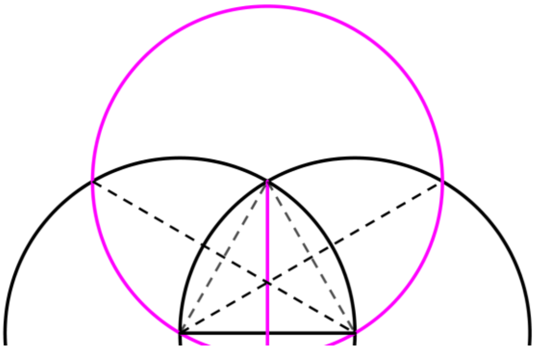
The perpendicular bisector construction itself is problematic in that the purposeful absence of circles in favour of arcs makes intuitive understanding far more elusive. With the addition of full circles and radii, you can now see the construction of several isosceles triangles:



As each congruent pair of isosceles triangles shares a base, the line joining their apexes *must* be perpendicular, otherwise they would not be isosceles.

There are in fact a number of ways to construct a perpendicular bisector, a few of which we shall explore here. The first is an extension of the isosceles triangles with a shared base idea. In fact, as long as the midpoint of their shared base is the same, the triangles need not be congruent:

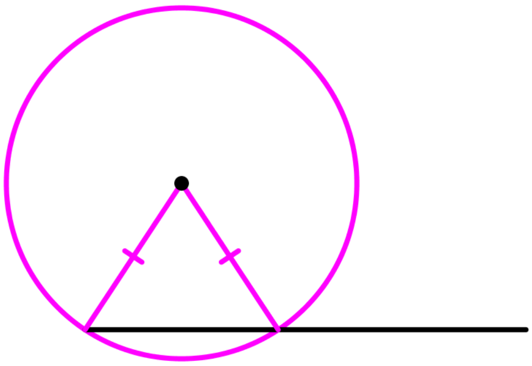




Of course, these approaches both require more construction lines, but they do have an advantage in that they require nothing to go *below* the line. Hence you can construct to the bottom of a page rather than the middle.

Perpendicular Line from a Point

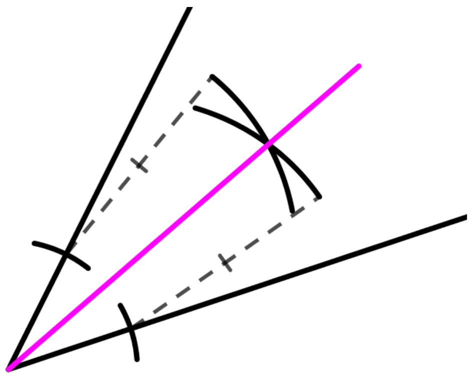
Construct a circle such that the center is the point and the circle intersects the line:

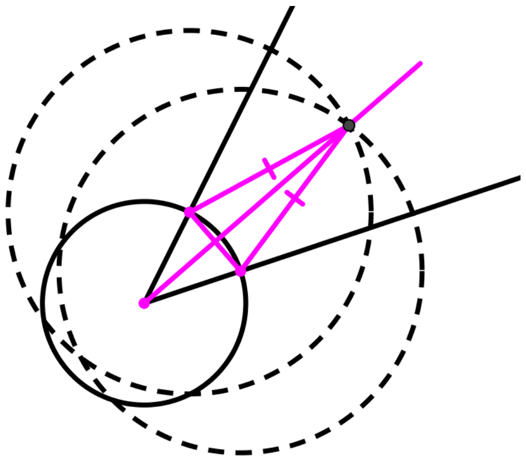


From the diagram, we have formed an isosceles triangle between the radii and the point, and therefore we are back to using our perpendicular bisector techniques.

Angle Bisection

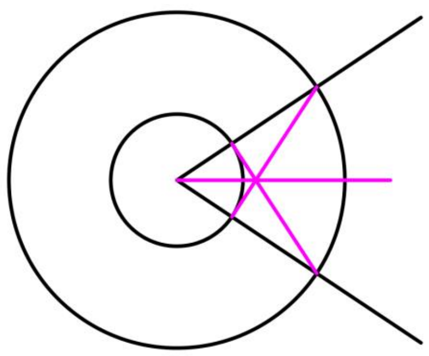
Much like with a perpendicular bisector, by drawing full circles instead of short arcs, previously obstructed conceptual understanding will become clearer. Techniques here are much the same as perpendicular bisectors - in that a straight line from the midpoint of the base of the isosceles triangle joined to the apex is not only perpendicular, but it also bisects the angle at the apex. This can be seen in the traditional method below.





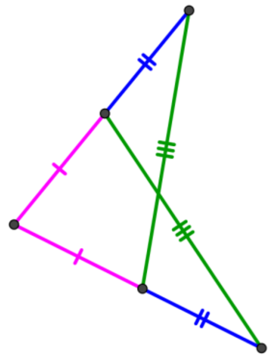
As always there are a number of alternative approaches should you want to explore further.

The method below was shown to me by Don Steward, a mathematician and retired teacher:

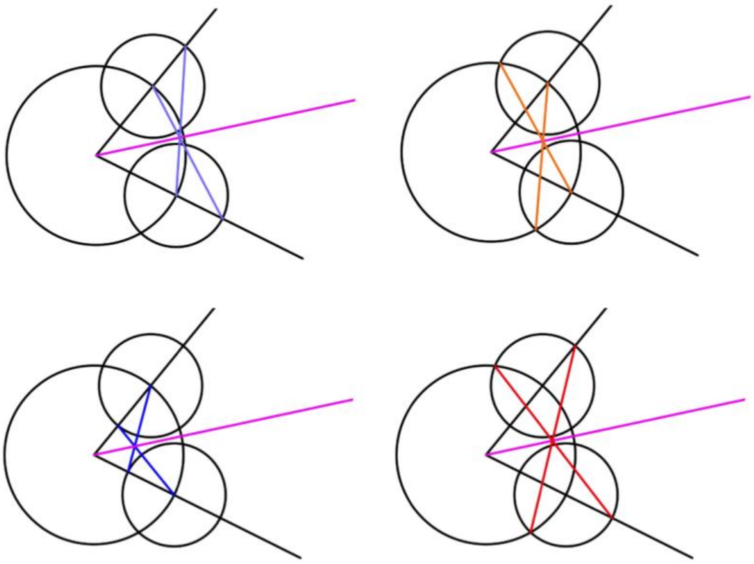


A proof of this method was submitted to Maths in Schools by Damian Griffin (1978).

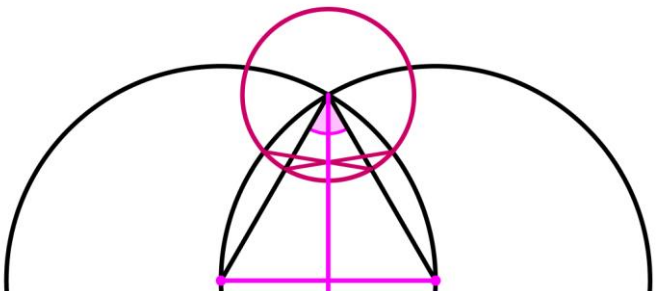
Using the principles of this technique, it becomes apparent that as long as you have a generalised structure like this:



then you will bisect the angle. This realisation opens up a wealth of alternative approaches - each subtly different but fundamentally the same mathematically, such as:



Indeed, we can use any angle bisection method to find the perpendicular bisector too:



References

Griffin, D. (1978). Bisecting an Angle. *Mathematics in School, 7*(5), 11-11. Retrieved from http://www.jstor.org/stable/30213417