

Hesitant fuzzy power Maclaurin symmetric mean operators in the framework of Dempster-Shafer theory for multiple criteria decision making

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Abstract

In this paper, a hesitant fuzzy power Maclaurin symmetric mean operator and a hesitant fuzzy weighted power Maclaurin symmetric mean operator are presented. The properties of these operators are explored and proved, and their special cases are discussed. Based on the presented operators, a new method is proposed to solve the multiple criteria decision making problems with hesitant fuzzy numbers. A numerical example is introduced to elucidate the application of the method. The advantages of the method are demonstrated via comparisons with some of the existing methods under a set of numerical examples. The demonstration results show that the proposed method has the capabilities to reduce the influence of biased evaluation values and consider the interrelationships between criteria, and more importantly it is free of the limitations of conventional operational laws of hesitant fuzzy numbers and applicable for the multiple criteria decision making problems where criterion weights are in the form of hesitant fuzzy numbers.

Keywords

Hesitant fuzzy number · Dempster-Shafer theory · Multiple criteria decision making · Power average operator · Maclaurin symmetric mean operator · Jusselme distance

1 Introduction

In practical multiple criteria decision making (MCDM) problems, it is normally difficult to use crisp numbers to express the values of criteria of alternatives. The fuzzy set theory was first proposed by Zadeh (1965), which is an effective tool for dealing with vague information. However, it is insufficient to deal with the information with non-membership degrees. Hesitant fuzzy sets (HFSs), which was proposed by Torra and Narukawa (2009) and Torra (2010), allow the membership degrees having a set of possible values rather than a single one (Garg et al. 2018). In recent years, HFSs have been applied to many fields, such as decision making (Karamaz et al. 2020; Şahin et al. 2020; Ullah et al. 2020; Zhang et al. 2019), clustering analysis (Karaaslan et al. 2020; Ullah et al. 2020), feature selection (Mokhtia et al. 2020), text classification (Lu et al. 2019), medicine selection (Ren et al. 2020), engineering management (Meng et al. 2020), and multi-objective programming problem (Rouhbakhsh et al. 2020). Recently, a large amount of research work has focused on the decision-making techniques based on HFSs, which comes from the following three areas:

(1) Theoretical foundations, such as operational rules (Torra and Narukawa 2009; Torra 2010; Wei et al. 2013), information measures (Farhadinia et al. 2013), score function (Farhadinia et al. 2013), similarity measures (Xu et al. 2011a; Xu et al. 2011b; Xu et al. 2012; Liu et al. 2020d;), correlation coefficients (Chen et al. 2013b), and distance measures (Xu et al. 2011a; Peng et al. 2013; Peng et al. 2020; Garg et al. 2020c).

(2) Conventional MCDM approaches based on HFSs, such as VIKOR-based methods (Liao et al. 2013; Zhang et al. 2013c; Chen et al. 2018; Liu et al. 2020f), TOPSIS-based methods (Lai et al. 1994; Dymova et al. 2013; Garg et al. 2020b), and HF-ELECTRE II-based methods (Chen et al. 2015).

(3) Aggregation operators for HFSs, such as the power average (PA) operator (Zhang 2013a), geometric Bonferroni mean operator (Xu et al. 2011c; He et al. 2015), Heronian mean operator (Liu et al. 2017), Dombi operator (Jana et al.

2019), Maclaurin symmetric mean (MSM) operator (Qin et al. 2014), probabilistic linguistic generalized Maclaurin symmetric mean operator (Liu et al. 2019b), and linguistic Archimedean aggregation operator (Liu et al. 2019c).

In general, the ranking results of the MCDM methods based on aggregation operators are better than that of the traditional MCDM methods in terms of credibility and accuracy, because aggregation operators can get integrated value of alternatives by aggregating all criterion values. Thus, it is of necessity and importance to construct the aggregation operators in MCDM environment to solve MCDM problems (Gao et al. 2019; Qin et al. 2019).

The PA operator was first introduced by Yager (2001), which can reduce the influence of unreasonable data on ranking results by measuring the similarity between arguments and assigning smaller weights to the arguments that are too high or too low. This advantageous characteristic is very useful in practical MCDM problems (Qin et al. 2020a). Therefore, researchers paid great attention to this operator. Xu and Yager (2010) proposed a geometric power average operator and power ordered weighted average operator on the basis of the PA operator. The PA operator has been applied to HFSs: Zhang (2013b) developed a hesitant fuzzy PA operator and a hesitant fuzzy power geometric operator; Zhu (2016) extended the PA operator to aggregate linguistic hesitant fuzzy numbers and investigated linguistic hesitant fuzzy power aggregation operators; Wang et al. (2016) developed some dual hesitant fuzzy power aggregation operators based on Archimedean t-conorm and t-norm.

In some MCDM problems, the interrelationships between criteria should be considered to obtain convincing aggregation results. The MSM operator (Maclaurin 1730) is one of the several aggregation operators with such capability (Liu et al. 2019b; Liu et al. 2020a; Liu et al. 2020b; Liu et al. 2020d; Qin et al. 2020b). In recent years, the extension of the MSM operator has attracted the attention of many researchers. Liu et al. (2019b) proposed some generalized Maclaurin symmetric mean operator for probabilistic linguistic information. Liu et al. (2020d) presented a linguistic neutrosophic partitioned Maclaurin symmetric mean operator and a linguistic neutrosophic weighted partitioned Maclaurin symmetric mean operator based on clustering algorithm. The MSM operator has also been extended to HFSs: Qin and Liu (2015) proposed a hesitant fuzzy MSM operator and a weighted hesitant fuzzy MSM operator. Garg and Arora (2020a) developed an MSM operator based on t-norm operations.

In an aggregation operator for HFSs, the aggregation results are generally calculated using operational laws of hesitant fuzzy numbers (HFNs). Torra and Narukawa (2009) and Torra (2010) proved that the operations on HFSs and HFNs are consistent with those of intuitionistic fuzzy sets (Atanassov 1986) when they are applied to the envelope of HFSs. Therefore, the operational laws for intuitionistic fuzzy sets are currently used for different problems in the framework of HFSs (Wei et al. 2013; Zhang et al. 2013b; Zhang et al. 2013c; Chen et al. 2013a; Qian et al. 2013; Wei 2012; Zhu et al. 2012). However, it was shown in (Dymova et al. 2010; Dymova et al. 2012; Dymova et al. 2016) that the ordinary operational laws have some undesirable properties and may provide counterintuitive results. It is obvious that the operations on HFNs defined in (Torra and Narukawa 2009; Torra 2010) have the same undesirable properties and could provide unreasonable solutions for MCDM problems.

Dempster-Shafer Theory (DST), which was first proposed by Dempster (1967) and then expanded by Shafer (1976), can successfully handle vague information. In the framework of DST, the incidence of criteria is represented by basic probability assignment (BPA), and the belief function and plausibility function constitute the belief interval (BI) that indicates the belief and uncertainty of focal element. As described in (Sevastjanov et al. 2015), there exists a close relationship between HFSs and DST, which allows criterion values of alternatives represented by HFNs to be directly aggregated by the aggregation laws of DST in an actual MCDM problem. Hence, in the framework of DST, a BI can be regarded as an envelope of HFNs and a BPA can rewrite an HFN without losing relevant information, the operations on HFNs can be expressed as the operations on BIs, and the Jousselme distance can be used to replace the Hamming distance and Euclidean distance in the calculation. These can overcome the shortcomings revealed by the ordinary operational laws of HFNs, and obtain more accurate and credible aggregation results (Liu et al. 2018; Liu et al. 2019a; Liu et al. 2020c; Qin et al. 2020c).

As described above, in addition to the undesirable properties and interrelationships of criteria, the actual MCDM problems could encounter some extreme evaluation values. However, there are few aggregation operators for HFSs that

are free of the undesirable properties and can reduce the influence of extreme evaluation values and simultaneously capture the interrelationships between criteria. According to the characteristics of the operational laws of HFNs in the framework of DST and the PA and MSM operators, it is obvious that a combination of them can meet these requirements. Based on this consideration, the objectives of this paper are summarised as follows:

(1) To present a hesitant fuzzy power MSM ($HFPM_{DST}$) operator and a hesitant fuzzy weighted power MSM ($HFWPMSM_{DST}$) operator in the framework of DST. The operators can not only get rid of the shortcomings of the existing aggregation operators of HFNs, but also reduce the influence of extreme evaluation values and capture the interrelationships among multiple criteria.

(2) To propose a new MCDM method based on the presented operators.

(3) To illustrate the advantages of the proposed method by comparing it with some existing methods from several aspects.

The rest of the paper is organised as follows: Sect. 2 briefly introduces some basic concepts of HFSs, analyzes the shortcomings of ordinary operational laws for HFSs, and gives an explanation of HFSs from the perspective of DST. Sect. 3 proposes the $HFPM_{DST}$ and $HFWPMSM_{DST}$ operators. Sect. 4 develops a new MCDM method with HFNs based on the $HFWPMSM_{DST}$ operator. Sect. 5 uses some practical examples to illustrate and demonstrate the proposed operators. Sect. 6 documents a conclusion.

2 Preliminaries

This section briefly reviews some basic concepts of HFSs, ordinary operational laws of HFNs and their limitations, basic principle of DST, and definition of a power MSM (PMSM) operator.

2.1. Hesitant fuzzy sets

Definition 2.1 (Torra and Narukawa 2009; Torra 2010). Let X be a universe of discourse. A HFS on X is in terms of a function that returns a subset of $[0,1]$ when it is applied to X . For convenience, an HFS can be notated as:

$$E = \{ \langle x, h_E(x) \rangle \mid x \in X \}, \quad (1)$$

where $h_E(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set X .

For convenience, $h = h_E(x)$ is called a HFN and H is a set of all HFNs.

The envelope of HFN is IFN, which was proved by Torra and Narukawa (2009) and Torra (2010), the definition is as follows:

Definition 2.2 (Torra and Narukawa 2009; Torra 2010). Given an HFN h , we define an IFN $A_{env}(h)$ as the envelope of h , where $A_{env}(h)$ can be represented as $\langle h^-, 1 - h^+ \rangle$, ($h^- = \min\{\gamma \mid \gamma \in h\}$, $h^+ = \max\{\gamma \mid \gamma \in h\}$).

Base on the relationship between HFSs and IFSs, Xia and Xu (2011) introduced the operational laws on HFNs:

Definition 2.3 (Xia and Xu 2011). Let h , h_1 and h_2 be three HFNs, then

$$h^\lambda = \cup_{\gamma \in h} \{\gamma^\lambda\}, \quad \lambda > 0, \quad (2)$$

$$\lambda h = \cup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}, \quad \lambda > 0, \quad (3)$$

$$h_1 \oplus h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}, \quad (4)$$

$$h_1 \otimes h_2 = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}, \quad (5)$$

$$h_1 \cup h_2 = \cup_{\gamma \in h_1, \gamma \in h_2} \max\{\gamma_1, \gamma_2\}, \quad (6)$$

$$h_1 \cap h_2 = \cap_{\gamma \in h_1, \gamma \in h_2} \min\{\gamma_1, \gamma_2\}, \quad (7)$$

A hesitant fuzzy weighted arithmetic mean (HFWAM) operator and a hesitant fuzzy weighted geometric mean (HFWGM) operator were derived from the operational laws above.

Definition 2.4 (Xia and Xu 2011). Let h_j ($j = 1, 2, 3, \dots, n$) be a collection of HFNs. A HFWAM operator is a mapping $H^n \rightarrow H$ such that

$$HFWAM(h_1, h_2, \dots, h_n) = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_n \in h_n} \left\{ 1 - \prod_{j=1}^n (1 - \gamma_j)^{w_j} \right\}, \quad (8)$$

where $w = (w_1, w_2, \dots, w_{n-1}, w_n)^T$ is the weight vector of h_j ($j = 1, 2, 3, \dots, n$) with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

Definition 2.5 (Xia and Xu 2011). Let h_j ($j = 1, 2, 3, \dots, n$) be a collection of HFNs. A HFWGM operator is a mapping $H^n \rightarrow H$ such that:

$$HFWGM(h_1, h_2, \dots, h_n) = \cup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \dots, \gamma_n \in h_n} \left\{ \gamma_j^{w_j} \right\}, \quad (9)$$

where $w = (w_1, w_2, \dots, w_{n-1}, w_n)^T$ is the weight vector of h_j ($j = 1, 2, 3, \dots, n$) with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

To compare HFNs, Xia and Xu (2011) proposed the following comparison law:

Definition 2.6 (Xia and Xu 2011). For an HFN h , its score can be given by:

$$S(h) = \frac{1}{\#h} \sum_{\gamma \in h} \gamma, \quad (10)$$

where $\#h$ is the number of elements in h .

For HFNs h_1 and h_2 , the order relation between h_1 and h_2 is obtained as follows:

- (1) If $S(h_1) > S(h_2)$, then $h_1 > h_2$;
- (2) If $S(h_1) = S(h_2)$, then $h_1 = h_2$;
- (3) If $S(h_1) < S(h_2)$, then $h_1 < h_2$;

Let h, h_1, h_2 be three HFNs. Eqs. (2)-(5) can be directly proved to satisfy the following transformations:

$$h_1 \oplus h_2 = h_2 \oplus h_1, \quad (11)$$

$$h_1 \otimes h_2 = h_2 \otimes h_1, \quad (12)$$

$$\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2, \quad \lambda > 0 \quad (13)$$

$$(\lambda + \mu)h = \lambda h + \mu h, \quad \lambda, \mu > 0 \quad (14)$$

$$(h_1 \otimes h_2)^\lambda = h_1^\lambda \otimes h_2^\lambda, \quad \lambda > 0 \quad (15)$$

$$h^\lambda \otimes h^\mu = h^{\lambda + \mu}, \quad \lambda, \mu > 0 \quad (16)$$

In (Sevastjanov et al. 2015), it was demonstrated that Eqs. (2)-(10) could produce the irrational results due to some unreasonable properties when using the operators to solve MCDM problems:

(1) An important limitation of Eqs. (2)-(7) is that they do not include the operation $h_1^{h_2}$ (h_1 and h_2 are HFNs) which is often needed in MCDM problems using the HFWGM operator.

(2) Eq. (3) and Eq. (5) have an obvious shortcoming. It is not difficult to see that Eq. (3) is not an asymptotical case of Eq. (5) when h_1 is converted to the real value $\lambda > 0$.

(3) Eq. (3) is not an identity under multiplication. That is, $h_1 < h_2$ does not always generate $\lambda h_1 < \lambda h_2$ ($\lambda > 0$).

(4) Eq. (4) is not a constant operation. That is, $h_1 < h_2$ does not always generate $(h_1 \oplus h_3) < (h_2 \oplus h_3)$.

Some intuitive examples for these unreasonable properties are as follows:

Example 2.1. Let $h_1 = \{0.4, 0.6\}$ and $h_2 = \{0.5001\}$ be HFNs, and $\lambda = 0.5001$. It is easy to obtain that $h_1 \otimes h_2 = \{0.20004, 0.30006\}$, $\lambda h_1 = \{0.22544, 0.3760\}$, and $h_1 \otimes h_2 \neq \lambda h_1$.

Example 2.2. Let $h_1 = \{0.4, 0.6\}$ and $h_2 = \{0.5001\}$ be HFNs and $\lambda = 0.5$, then according to the Definition 2.6 we have $S(h_1) = 0.5, S(h_2) = 0.5001$ and therefore $h_2 > h_1$. However, $\lambda h_1 = \{0.22540, 0.3675\}$, $\lambda h_2 = \{0.29296\}$, $S(\lambda h_1) = 0.29647$, $S(\lambda h_2) = 0.29296$, and it is easier to obtain that $\lambda h_1 > \lambda h_2$.

Example 2.3. Let $h_1 = \{0.4, 0.6\}$, $h_2 = \{0.5001\}$ and $h_3 = \{0.3, 0.7\}$ be HFNs. Then $S(h_1) = 0.5, S(h_2) = 0.5001$, and it is easy to obtain that $h_1 < h_2$. However, given that $h_1 \oplus h_3 = \{0.58, 0.89, 0.72, 0.88\}$, $h_2 \oplus h_3 = \{0.65007, 0.85003\}$, $S(h_1 \oplus h_3) = 0.7675$, and $S(h_2 \oplus h_3) = 0.75005$, it is easier to obtain that $(h_1 \oplus h_3) > (h_2 \oplus h_3)$.

Additionally, Sevastjanov et al. (2015) showed that HFS is closely related to the DST, and this relationship can provide transparent and efficient semantics for HFS from the perspective of DST. Therefore, in the framework of DST, the purpose of enhancing the operational laws of HFNs and getting rid of the limitations listed above can be achieved by rewriting the definition and operations of HFS.

2.2. HFS in the framework of DST

Let the set of n exhaustive and mutually exclusive objects be θ , which corresponds to n propositions or hypotheses. θ is called the discrimination framework and defined as $\theta = \{1, 2, 3, \dots, m\}$. The power set of θ is $\Gamma(\theta)$, containing all the possible subsets of θ : $\Gamma(\theta) = \{\phi, 1, 2, 3, \dots, m, (1, 2), (1, 3), \dots, (m-1, m), (1, 2, 3), \dots, \theta\}$. According to the definition, the event “the object is in X ” is represented by $\Gamma(\theta)$.

Assume that a mapping from $\Gamma(\theta)$ to $[0, 1]$ is a BPA s that satisfies the two conditions of $\sum_{X \in \Gamma(\theta)} s(X) = 1$ and $s(\phi) = 0$, and $s(X)$ represents the degree of evidence support for the statement that the object belongs to X .

Definition 2.7 (Dempster 1967; Shafer 1976). Suppose there is a BPA s on θ , the definition of the belief function Bel can be expressed as follows:

$$Bel(X) = \sum_{Y \subseteq X} s(Y), \quad (17)$$

where $s(X) > 0$.

Definition 2.8 (Dempster 1967; Shafer 1976). Suppose there is a BPA s on θ , the definition of the plausibility function Pl can be expressed as follows:

$$Pl(X) = \sum_{Y \cap X \neq \phi} s(Y) = 1 - Bel(\bar{Y}), \quad (18)$$

where \bar{Y} is the complementary set of Y .

Therefore, $[Bel(X), Pl(X)]$ can represent the BI. This can be interpreted as the interval enclosing the “true, probability” of X .

To measure the similarity between two bodies of evidence, the Jousselme distance is described as follows:

Definition 2.9 (Dempster 1967; Shafer 1976). Let θ be a frame of discernment including n mutually exclusive and exhaustive hypothesis, and $\Lambda_{\Gamma(\theta)}$ be the space produced by all the subsets of θ . A BPA is a vector \vec{s} of $\Lambda_{\Gamma(\theta)}$ with coordinates $s(X_i)$ such that:

$$\sum_{i=1}^{2^n} s(X_i) = 1, \quad (19)$$

where $s(X_i) \geq 0, X_i \in \Gamma(\theta)$.

In the above definition, $s(X_i) = 0$ is not a necessary condition.

Definition 2.10 (Jousselme et al. 2001). Let s_1 and s_2 be two BPAs on the same frame of discernment θ , including n mutually exclusive and exhaustive hypotheses. The Jousselme distance between s_1 and s_2 can be defined as follows:

$$d_{BPA}(s_1, s_2) = \sqrt{\frac{1}{2} (\vec{s}_1 - \vec{s}_2)^T \underline{D} (\vec{s}_1 - \vec{s}_2)}, \quad (20)$$

where \vec{s}_1 and \vec{s}_2 are the BPAs according to Definition 2.9 and \underline{D} corresponds to an $2^n * 2^n$ matrix whose elements are as follows:

$$D(X, Y) = \frac{|X \cap Y|}{|X \cup Y|}, X, Y \in \Gamma(\theta). \quad (21)$$

From Definition 2.10, another description of d_{BPA} is as follows:

$$d_{BPA}(s_1, s_2) = \sqrt{\frac{1}{2} (\|\vec{s}_1\|^2 + \|\vec{s}_2\|^2 - 2 \langle \vec{s}_1, \vec{s}_2 \rangle)}, \quad (22)$$

where $\langle \vec{s}_1, \vec{s}_2 \rangle$ corresponds to the scalar product defined by

$$\langle \vec{s}_1, \vec{s}_2 \rangle = \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} s_1(X_i) s_2(X_j) \frac{|X_i \cap X_j|}{|X_i \cup X_j|}, \quad (23)$$

where $X_i, X_j \in \Gamma(\theta)$ for $i, j = 1, 2, \dots, 2^n$. $\|\vec{s}\|^2$ is the square norm of \vec{s} :

$$\|\vec{s}\|^2 = \langle \vec{s}, \vec{s} \rangle. \quad (24)$$

It has been proved in (Sevastjanov et al. 2015) that a BI is an envelope of an HFS. Therefore, it is not difficult to obtain the results which will be free of the drawbacks of known approaches using BIs as envelopes of HFSs.

Let $h = \{e_1, e_2, \dots, e_n\}$ be an HFN, then $\underline{h} = \min\{e_1, e_2, \dots, e_n\}$, $\bar{h} = \max\{e_1, e_2, \dots, e_n\}$ and $\tilde{h} = (1 - \underline{h} - \bar{h})$ of HFN h can represent a BPA of DST (Sevastjanov et al. 2015), and thus the aforementioned information on DST can be used to represent the HFS. In practical MCDM problems, when using HFNs to solve decision problems, three hypotheses are clandestinely faced as follows: True, False, True or False (the case of hesitation). In this situation, the lowest estimate $\underline{h} = \min\{e_1, e_2, \dots, e_n\}$ as a more guaranteed (reliable) one as it is the pessimistic estimation, hence $s(True) = \underline{h}$. On the other hand, the value $\bar{h} = \max\{e_1, e_2, \dots, e_n\}$ may be treated only as the plausible one as this is the maximal estimation, hence $s(True) + s(True or False) = \bar{h}$. By analogy, $s(False) = 1 - \bar{h}$. Eventually, since $\langle \underline{h}, 1 - \bar{h} \rangle$ is IF envelope of the HFS H , it can be represented in the framework of DST as $H_{DST} = [Bel, Pl]$, where Bel and Pl are given below:

$$Bel = s(True) = \underline{h}, (\underline{h} = \min\{e_1, e_2, \dots, e_n\}); \quad (25)$$

$$Pl = s(True) + s(True + False) = \bar{h}, (\bar{h} = \max\{e_1, e_2, \dots, e_n\}). \quad (26)$$

Therefore, an HFS in the framework of DST is redefined as follows:

Definition 2.11 (Sevastjanov et al. 2015). Let $X = \{x_i | i = 1, 2, \dots, m\}$ be a fixed set, then an HFS H on X in the framework of DST is defined as $H_{DST} = \{x_i, \tilde{h}_B(x_i), i = 1, 2, \dots, n\}$, where $\tilde{h}_B(x_i)$ denotes the set of BIs (including degenerated BIs, i.e., real values) representing the membership degrees of x_i to X .

To simplify the following formula, an HFN in the framework of DST is expressed by $\tilde{h} = [Bel_i, Pl_i] = (\underline{h}_i, \bar{h}_i)$, where $\underline{h} = \min\{e_1, e_2, \dots, e_n\}$ and $\bar{h} = \max\{e_1, e_2, \dots, e_n\}$.

As the envelope of HFS is BI, the operational laws on HFNs were redefined in the framework of DST by Sevastjanov and Dymova (2015).

Definition 2.12 (Sevastjanov et al. 2015). Let $\tilde{h}_1 = [Bel_1, Pl_1] = (\underline{h}_1, \bar{h}_1)$ and $\tilde{h}_2 = [Bel_2, Pl_2] = (\underline{h}_2, \bar{h}_2)$ be two HFNs in the framework of DST, then the expression is as follows:

$$\tilde{h}_1 \oplus \tilde{h}_2 = \left[\frac{Bel_1 + Bel_2}{2}, \frac{Pl_1 + Pl_2}{2} \right] = \left[\frac{\underline{h}_1 + \underline{h}_2}{2}, \frac{\bar{h}_1 + \bar{h}_2}{2} \right]; \quad (27)$$

$$\tilde{h}_1 \otimes \tilde{h}_2 = [Bel_1 Bel_2, Pl_1 Pl_2] = [\underline{h}_1 \underline{h}_2, \bar{h}_1 \bar{h}_2]; \quad (28)$$

$$\lambda \tilde{h}_1 = [\lambda Bel_1, \lambda Pl_1] = [\lambda \underline{h}_1, \lambda \bar{h}_1], \lambda > 0; \quad (29)$$

$$\tilde{h}_1^\lambda = [Bel_1^\lambda, Pl_1^\lambda] = [\underline{h}_1^\lambda, \bar{h}_1^\lambda]; \quad (30)$$

$$\tilde{h}_1^{\tilde{h}_2} = [Bel_1, Pl_1]^{[Bel_2, Pl_2]} = [Bel_1^{Pl_2}, Pl_1^{Bel_2}] = [\underline{h}_1^{\bar{h}_2}, \bar{h}_1^{\underline{h}_2}]; \quad (31)$$

$$\frac{\tilde{h}_1}{\tilde{h}_2} = \frac{[Bel_1, Pl_1]}{[Bel_2, Pl_2]} = \left[\frac{Bel_1}{Pl_2}, \frac{Pl_1}{Bel_2} \right] = \left[\frac{\underline{h}_1}{\bar{h}_2}, \frac{\bar{h}_1}{\underline{h}_2} \right]; \quad (32)$$

From the aforementioned operational laws, an $HFWM_{DST}$ operator and an $HFWM_{DST}$ operator are obtained in the framework of DST. Let $\tilde{h}_i = [Bel_i, Pl_i] = [\underline{h}_i, \bar{h}_i] (i = 1, 2, \dots, n)$ be HFNs in the framework of DST, representing the values of local criteria, w_i be real valued weights of these criteria such that $\sum_{i=1}^n w_i = 1$. Then the $HFWM_{DST}$ operator is given by:

$$HFWAM_{DST}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \left[\frac{1}{n} \sum_{i=1}^n w_i Bel_i, \frac{1}{n} \sum_{i=1}^n w_i Pl_i \right] = \left[\frac{1}{n} \sum_{i=1}^n w_i \underline{h}_i, \frac{1}{n} \sum_{i=1}^n w_i \bar{h}_i \right]. \quad (33)$$

The $HFWGM_{DST}$ operator is given by:

$$HFWGM_{DST}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_n) = \left[\prod_{i=1}^n Bel_i^{w_i}, \prod_{i=1}^n Pl_i^{w_i} \right] = \left[\prod_{i=1}^n \underline{h}_i^{w_i}, \prod_{i=1}^n \bar{h}_i^{w_i} \right]. \quad (34)$$

To compare HFNs in the framework of DST, the definitions of score function (SF_{DST}) and accuracy function (AF_{DST}) are as follows:

$$SF_{DST}(\tilde{h}) = \frac{Bel + Pl}{2}, [Bel, Pl] \subseteq \tilde{h}. \quad (35)$$

$$AF_{DST}(\tilde{h}) = Pl - Bel, [Bel, Pl] \subseteq \tilde{h}. \quad (36)$$

In addition, the order relation between \tilde{h}_1 and \tilde{h}_2 is given as follows:

- (1) If $SF_{DST}(\tilde{h}_1) < SF_{DST}(\tilde{h}_2)$, then $\tilde{h}_1 < \tilde{h}_2$;
- (2) If $SF_{DST}(\tilde{h}_1) = SF_{DST}(\tilde{h}_2)$, then
 - (i) If $AF_{DST}(\tilde{h}_1) > AF_{DST}(\tilde{h}_2)$, then $\tilde{h}_1 < \tilde{h}_2$;
 - (ii) If $AF_{DST}(\tilde{h}_1) = AF_{DST}(\tilde{h}_2)$, then $\tilde{h}_1 = \tilde{h}_2$;

It is not difficult to see that the operational law in Eq. (10) is closely related to the operational laws in Eqs. (35) and (36). However, it is not suitable to apply the operational law in Eq. (10) to compare BIs and apply the operational laws in Eqs. (35) and (36) to compare HFNs.

Let \tilde{h} , \tilde{h}_1 and \tilde{h}_2 be three BIs and λ, μ, τ be real values. It is not difficult to prove that Eqs. (27)-(32) have the following properties (Sevastjanov et al. 2015):

$$\tilde{h}_1 \oplus \tilde{h}_2 = \tilde{h}_2 \oplus \tilde{h}_1; \quad (37)$$

$$\tilde{h}_1 \otimes \tilde{h}_2 = \tilde{h}_2 \otimes \tilde{h}_1; \quad (38)$$

$$\lambda(\tilde{h}_1 \oplus \tilde{h}_2) = \lambda \tilde{h}_1 \oplus \lambda \tilde{h}_2, \quad \lambda > 0; \quad (39)$$

$$(\mu + \tau)\tilde{h} = \mu\tilde{h} + \tau\tilde{h}, \quad \mu, \tau > 0; \quad (40)$$

$$(\tilde{h}_1 \otimes \tilde{h}_2)^\lambda = \tilde{h}_1^\lambda \otimes \tilde{h}_2^\lambda, \quad \lambda > 0; \quad (41)$$

$$\tilde{h}^\mu \otimes \tilde{h}^\tau = \tilde{h}^{\mu+\tau}, \quad \mu, \tau > 0; \quad (42)$$

As demonstrated in (Sevastjanov et al. 2015), the above operational laws of HFNs in the framework of DST can overcome the drawbacks of the traditional operational laws of HFNs in Eqs. (11)-(16).

2.3. PMSM operator

The PA operator was first proposed by Yager (2001), which requires the criteria values to support and balance each other in the process of aggregating information, and the MSM operator was first introduced by Maclaurin (1730), which captures the interrelationships between multiple input criteria parameters by changing a variable κ .

Definition 2.13 (Yager 2001). Let $\chi_\eta (\eta = 1, 2, \dots, t)$ be a set of evaluated values, the definition of the PA operator of $\chi_\eta (\eta = 1, 2, \dots, t)$ is as follows:

$$PA(\chi_1, \chi_2, \dots, \chi_t) = \frac{\sum_{\eta=1}^t (1 + T(\chi_\eta)) \chi_\eta}{\sum_{\eta=1}^t (1 + T(\chi_\eta))}, \quad (43)$$

where $T(\chi_\eta) = \sum_{\tau=1, \tau \neq \eta}^t Sup(\chi_\eta, \chi_\tau)$, $Sup(\chi_\eta, \chi_\tau) = 1 - d(\chi_\eta, \chi_\tau)$, and $Sup(\chi_\eta, \chi_\tau)$ represents the level of support from χ_τ to χ_η , which satisfies the following conditions:

- (1) $Sup(\chi_\eta, \chi_\tau) \in [0, 1]$;
- (2) $Sup(\chi_\eta, \chi_\tau) = Sup(\chi_\tau, \chi_\eta)$;

(3) $Sup(\chi_\eta, \chi_\tau) > Sup(\chi_k, \chi_l)$, if $d(\chi_\eta, \chi_\tau) < d(\chi_k, \chi_l)$.

where $\eta, \tau, k, l \in \{1, 2, \dots, t\}$.

Definition 2.14 (Maclaurin 1730). Let $\chi_\eta (\eta = 1, 2, \dots, t)$ be a set of positive numbers, the definition of the MSM operator of $\chi_\eta (\eta = 1, 2, \dots, t)$ is as follows:

$$MSM^{(\kappa)}(\chi_1, \chi_2, \dots, \chi_t) = \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_\kappa \leq t} \prod_{j=1}^{\kappa} \chi_{\eta_j}}{C_t^\kappa} \right)^{\frac{1}{\kappa}}, \quad (44)$$

where $C_t^\kappa = t!/\kappa!(t-\kappa)!$ is the binomial coefficient, and $(\eta_1, \eta_2, \dots, \eta_\kappa)$ traverses all the κ -tuple ($1 \leq \kappa \leq t$) combination of $(1, 2, \dots, t)$, which has the following properties:

(1) $MSM^{(\kappa)}(0, 0, \dots, 0) = 0$, $MSM^{(\kappa)}(\chi, \chi, \dots, \chi) = \chi$;

(2) $MSM^{(\kappa)}(\chi_1, \chi_2, \dots, \chi_t) \leq MSM^{(\kappa)}(\chi_1', \chi_2', \dots, \chi_t')$, $\chi_\eta \leq \chi_\eta'$ for all η ;

(3) $\min_\eta \{\chi_\eta\} \leq MSM^{(\kappa)}(\chi_1, \chi_2, \dots, \chi_t) \leq \max_\eta \{\chi_\eta\}$.

Definition 2.15 (Gao et al. 2019; Liu et al. 2020a). Let $\chi_\eta (\eta = 1, 2, \dots, t)$ be a set of positive numbers, the definition of the PMSM operator of $\chi_\eta (\eta = 1, 2, \dots, t)$ is as follows:

$$PMSM^\kappa(\chi_1, \chi_2, \dots, \chi_t) = \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_\kappa \leq t} \prod_{j=1}^{\kappa} \frac{t(1+T(\chi_{\eta_j}))\chi_{\eta_j}}{\sum_{\eta=1}^t (1+T(\chi_\eta))}}{C_t^\kappa} \right)^{\frac{1}{\kappa}}, \quad (45)$$

where $T(\chi_\eta) = \sum_{\tau=1, \tau \neq \eta}^t Sup(\chi_\eta, \chi_\tau)$, $Sup(\chi_\eta, \chi_\tau) = 1 - d(\chi_\eta, \chi_\tau)$, and $Sup(\chi_\eta, \chi_\tau)$ denotes the grade of support for χ_η from χ_τ , $C_t^\kappa = t!/\kappa!(t-\kappa)!$ is the binomial coefficient, and $(\eta_1, \eta_2, \dots, \eta_\kappa)$ traverses all the κ -tuple ($1 \leq \kappa \leq t$) combination of $(1, 2, \dots, t)$.

3 HFPMSM operator in the framework of DST

Based on DST, the PMSM operator is extended to HFNs, and the $HFPMSM_{DST}$ and $HFPWMSM_{DST}$ operators are proposed in this section.

3.1. $HFPMSM_{DST}$ operator

Definition 3.1. Let $\tilde{H} = \{\tilde{h}_\eta | \tilde{h}_\eta = [Bel_\eta, Pl_\eta], \eta = 1, 2, \dots, t\}$ be the corresponding BI set of HFS $H = \{h_\eta | h_\eta, \eta = 1, 2, \dots, t\}$. The $HFPMSM_{DST}$ operator of $\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t$ can be defined as follows:

$$HFPMSM_{DST}^\kappa(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) = \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_\kappa \leq t} \prod_{j=1}^{\kappa} \frac{t(1+T(\tilde{h}_{\eta_j}))\tilde{h}_{\eta_j}}{\sum_{\eta=1}^t (1+T(\tilde{h}_\eta))}}{C_t^\kappa} \right)^{\frac{1}{\kappa}}, \quad (46)$$

where $T(\tilde{h}_\eta) = \sum_{\tau=1, \tau \neq \eta}^t Sup(\tilde{h}_\eta, \tilde{h}_\tau)$, $Sup(\tilde{h}_\eta, \tilde{h}_\tau) = 1 - d(\tilde{h}_\eta, \tilde{h}_\tau)$, and $Sup(\tilde{h}_\eta, \tilde{h}_\tau)$ represents the level of support from \tilde{h}_τ to \tilde{h}_η , which has some properties that listed in Definition 2.13.

Theorem 3.1. Let $\tilde{H} = \{\tilde{h}_\eta | \tilde{h}_\eta = [Bel_\eta, Pl_\eta], \eta = 1, 2, \dots, t\}$ be the corresponding BI set of HFS $H = \{h_\eta | h_\eta, \eta = 1, 2, \dots, t\}$. The aggregate result calculated by the $HFPMSM_{DST}$ operator is still a BI, which can be expressed by:

$$= \left[\left(\frac{\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} \frac{t(1+T(\widetilde{h}_{\eta_j})) \text{Bel}_{\eta_j}}{\sum_{\eta=1}^t (1+T(\widetilde{h}_{\eta}))} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right), \left(\frac{\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} \frac{t(1+T(\widetilde{h}_{\eta_j})) \text{Pl}_{\eta_j}}{\sum_{\eta=1}^t (1+T(\widetilde{h}_{\eta}))} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right) \right].$$

For simplify the above operator, let $\theta_{\eta} = \frac{1+T(\widetilde{h}_{\eta})}{\sum_{\eta=1}^t (1+T(\widetilde{h}_{\eta}))}$, then, Eq. (47) is given as follows:

$$HFPMSTM_{DST}^{\kappa}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) = \left[\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Bel}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right), \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Pl}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right) \right], \quad (48)$$

Proof. We can use the operational laws of HFNs in the framework of DST to infer $t\theta_{\eta_j} \widetilde{h}_{\eta_j} = t\theta_{\eta_j} [\text{Bel}_{\eta_j}, \text{Pl}_{\eta_j}] = [t\theta_{\eta_j} \text{Bel}_{\eta_j}, t\theta_{\eta_j} \text{Pl}_{\eta_j}]$. Then, $\prod_{j=1}^{\kappa} t\theta_{\eta_j} \widetilde{h}_{\eta_j} = \prod_{j=1}^{\kappa} [t\theta_{\eta_j} \text{Bel}_{\eta_j}, t\theta_{\eta_j} \text{Pl}_{\eta_j}] = [\prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Bel}_{\eta_j}, \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Pl}_{\eta_j}]$.

Therefore:

$$\begin{aligned} \sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \widetilde{h}_{\eta_j} &= \sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} [\prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Bel}_{\eta_j}, \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Pl}_{\eta_j}], \\ &= \left[\frac{t^{\kappa} \sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} \theta_{\eta_j} \text{Bel}_{\eta_j}}{C_t^{\kappa}}, \frac{t^{\kappa} \sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} \theta_{\eta_j} \text{Pl}_{\eta_j}}{C_t^{\kappa}} \right]. \end{aligned}$$

Finally, the following expression can be obtained:

$$HFPMSTM_{DST}^{\kappa}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) = \left[\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Bel}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right), \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Pl}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right) \right].$$

Also, some ideal properties of the $HFPMSTM_{DST}$ operator are listed as follows:

Theorem 3.2 (Commutativity). Let $\widetilde{h}_{\eta} = [\text{Bel}_{\eta}, \text{Pl}_{\eta}] (\eta = 1, 2, \dots, t)$ be the corresponding BIs of HFNs $h_{\eta} (\eta = 1, 2, \dots, t)$. If $(\widetilde{h}'_1, \widetilde{h}'_2, \dots, \widetilde{h}'_t)$ is any permutation of $(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t)$, $\widetilde{h}'_{\eta} = [\text{Bel}'_{\eta}, \text{Pl}'_{\eta}]$, then $HFPMSTM_{DST}^{\kappa}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) = HFPMSTM_{DST}^{\kappa}(\widetilde{h}'_1, \widetilde{h}'_2, \dots, \widetilde{h}'_t)$.

Proof. Assume $(\widetilde{h}'_1, \widetilde{h}'_2, \dots, \widetilde{h}'_t)$ is any permutation of $(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t)$, based on Theorem 3.1, the following expression can be obtained:

$$\begin{aligned} HFPMSTM_{DST}^{\kappa}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) &= \left[\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Bel}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right), \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Pl}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right) \right] \\ &= \left[\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Bel}'_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right), \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Pl}'_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right) \right] \\ &= HFPMSTM_{DST}^{\kappa}(\widetilde{h}'_1, \widetilde{h}'_2, \dots, \widetilde{h}'_t). \end{aligned}$$

Theorem 3.3 (Boundedness). Let $\widetilde{h}_{\eta} = [\text{Bel}_{\eta}, \text{Pl}_{\eta}] (\eta = 1, 2, \dots, t)$ be the corresponding BIs of HFNs $h_{\eta} (\eta = 1, 2, \dots, t)$. If $\widetilde{h}_{\eta}^{+} = [\max_{\eta=1}^t (\text{Bel}_{\eta}), \max_{\eta=1}^t (\text{Pl}_{\eta})]$ and $\widetilde{h}_{\eta}^{-} = [\min_{\eta=1}^t (\text{Bel}_{\eta}), \min_{\eta=1}^t (\text{Pl}_{\eta})]$, then $\widetilde{h}_{\eta}^{-} \leq HFPMSTM_{DST}^{\kappa}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) \leq \widetilde{h}_{\eta}^{+}$.

Proof. Since $\theta_{\eta} = \frac{1+T(\widetilde{h}_{\eta})}{\sum_{\eta=1}^t (1+T(\widetilde{h}_{\eta}))}$ and $\sum_{\eta=1}^t \theta_{\eta} = 1$, it is easy to obtain that $\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Bel}_{\eta_j}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} \leq \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \max_{\eta=1}^t (\text{Bel}_{\eta_j})}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}}$ and $\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \text{Pl}_{\eta_j}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} \leq \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^{\kappa} t\theta_{\eta_j} \max_{\eta=1}^t (\text{Pl}_{\eta_j})}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}}$.

Therefore, based on the Eq. (35), the following expression can be obtained:

$$\begin{aligned}
SF\left(HFPMSM_{DST}^{\kappa}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t)\right) &= \frac{1}{2} \left(\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} Bel_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} + \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} Pl_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} \right) \\
&\leq \frac{1}{2} \left(\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} \max_{\eta=1}^t (Bel_{\eta_j}) \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} + \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} \max_{\eta=1}^t (Pl_{\eta_j}) \right)^{\frac{1}{\kappa}}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} \right) = SF(\tilde{h}_{\eta}^+).
\end{aligned}$$

That is

$$\begin{aligned}
HFPMSM_{DST}^{\kappa}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) &= \left[\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} Bel_{\eta_j}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}}, \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} Pl_{\eta_j}}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} \right] \\
&\leq \left[\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} \max_{\eta=1}^t (Bel_{\eta_j})}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}}, \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} \max_{\eta=1}^t (Pl_{\eta_j})}{(C_t^{\kappa})^2} \right)^{\frac{1}{\kappa}} \right] = \tilde{h}_{\eta}^+.
\end{aligned}$$

Similarly, it is not difficult to obtain that $\tilde{h}_{\eta}^- \leq HFPMSM_{DST}^{\kappa}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t)$. Therefore, it is obtained that:

$$\tilde{h}_{\eta}^- \leq HFPMSM_{DST}^{\kappa}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) \leq \tilde{h}_{\eta}^+.$$

It should also be noted that $HFPMSM_{DST}$ operator is not idempotent. However, if this operator is multiplied by $(C_t^{\kappa})^{\frac{1}{\kappa}}$, the multiplied $HFPMSM_{DST}$ ($MHFPMSM_{DST}$) operator corresponds to an idempotent operator as follows:

$$MHFPMSM_{DST}^{\kappa}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) = \left[\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} Bel_{\eta_j}}{C_t^{\kappa}} \right)^{\frac{1}{\kappa}}, \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\theta_{\eta_j} Pl_{\eta_j}}{C_t^{\kappa}} \right)^{\frac{1}{\kappa}} \right]. \quad (49)$$

Evidently, the $HFPMSM_{DST}$ and $MHFPMSM_{DST}$ operators can get the same ranking results of alternatives in a real decision-making problem. In addition, it should be noted that the $HFPMSM_{DST}$ operator is not monotonic, because d_{BPA} will change when the criterion values and parameter κ change.

3.2. $HFWPMSM_{DST}$ operator

The $HFPMSM_{DST}$ operator does not take the weights of the criteria into account. However, in real decision making problems, the weights of criteria are very important in the process of aggregating information. Therefore, the $HFWPMSM_{DST}$ operator that can consider the weight information of criteria is proposed.

Definition 3.2. Let $\tilde{H} = \{\tilde{h}_{\eta} | \tilde{h}_{\eta} = [Bel_{\eta}, Pl_{\eta}], \eta = 1, 2, \dots, t\}$ be the corresponding BI set of HFS $H = \{h_{\eta} | h_{\eta}, \eta = 1, 2, \dots, t\}$. ω_{η} denotes the weight of \tilde{h}_{η} such that $\sum_{\eta=1}^t \omega_{\eta} = 1$. The $HFWPMSM_{DST}$ operator of $\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t$ is defined as follows:

$$HFWPMSM_{DST}^{(\kappa)}(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) = \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k \frac{t\omega_{\eta_j} (1 + T(\tilde{h}_{\eta_j})) \tilde{h}_{\eta_j}}{\sum_{\eta=1}^t \omega_{\eta} (1 + T(\tilde{h}_{\eta}))}}{C_t^{\kappa}} \right)^{\frac{1}{\kappa}}, \quad (50)$$

where $T(\tilde{h}_{\eta}) = \sum_{\tau=1, \tau \neq \eta}^t Sup(\tilde{h}_{\eta}, \tilde{h}_{\tau})$, $Sup(\tilde{h}_{\eta}, \tilde{h}_{\tau}) = 1 - d(\tilde{h}_{\eta}, \tilde{h}_{\tau})$, and $Sup(\tilde{h}_{\eta}, \tilde{h}_{\tau})$ denotes the support degree for \tilde{h}_{η} from \tilde{h}_{τ} .

Theorem 3.4. Let $\tilde{H} = \{\tilde{h}_{\eta} | \tilde{h}_{\eta} = [Bel_{\eta}, Pl_{\eta}], \eta = 1, 2, \dots, t\}$ be the corresponding BI set of HFS $H = \{h_{\eta} | h_{\eta}, \eta = 1, 2, \dots, t\}$. The aggregate result calculated by the $HFWPMSM_{DST}$ operator is still a BI, which can be expressed by:

$$\begin{aligned}
& HFWPMSM_{DST}^\kappa(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) \tag{51} \\
& = \left[\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k \frac{t\omega_{\eta_j} (1 + T(\tilde{h}_{\eta_j})) \text{Bel}_{\eta_j}}{\sum_{\eta=1}^t \omega_{\eta} (1 + T(\tilde{h}_{\eta}))} \right)^{\frac{1}{\kappa}}}{(C_t^\kappa)^2} \right), \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k \frac{t\omega_{\eta_j} (1 + T(\tilde{h}_{\eta_j})) \text{Pl}_{\eta_j}}{\sum_{\eta=1}^t \omega_{\eta} (1 + T(\tilde{h}_{\eta}))} \right)^{\frac{1}{\kappa}}}{(C_t^\kappa)^2} \right) \right].
\end{aligned}$$

To simplify the above operator, let $\bar{\theta}_\eta = \frac{\omega_\eta(1+T(\tilde{h}_\eta))}{\sum_{\eta=1}^t \omega_\eta(1+T(\tilde{h}_\eta))}$, then Eq. (51) is given as follows:

$$\begin{aligned}
& HFWPMSM_{DST}^\kappa(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) \tag{52} \\
& = \left[\left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\bar{\theta}_{\eta_j} \text{Bel}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^\kappa)^2} \right), \left(\frac{\left(\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\bar{\theta}_{\eta_j} \text{Pl}_{\eta_j} \right)^{\frac{1}{\kappa}}}{(C_t^\kappa)^2} \right) \right].
\end{aligned}$$

Proof. The proof is basically consistent with Theorem 3.1, and it is omitted here.

Also, some ideal properties of the $HFWPMSM_{DST}$ operator are listed as follows:

Theorem 3.5 (Commutativity). Let $\tilde{h}_\eta = [\text{Bel}_\eta, \text{Pl}_\eta]$ ($\eta = 1, 2, \dots, t$) be the corresponding BIs of HFNs h_η ($\eta = 1, 2, \dots, t$). If $(\tilde{h}'_1, \tilde{h}'_2, \dots, \tilde{h}'_t)$ is any permutation of $(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t)$, then $HFWPMSM_{DST}^\kappa(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) = HFWPMSM_{DST}^\kappa(\tilde{h}'_1, \tilde{h}'_2, \dots, \tilde{h}'_t)$.

Theorem 3.6 (Boundedness). Let $\tilde{h}_\eta = [\text{Bel}_\eta, \text{Pl}_\eta]$ ($\eta = 1, 2, \dots, t$) be the corresponding BIs of HFNs h_η ($\eta = 1, 2, \dots, t$). If $\tilde{h}_\eta^+ = [\max_{\eta=1}^t(\text{Bel}_\eta), \max_{\eta=1}^t(\text{Pl}_\eta)]$ and $\tilde{h}_\eta^- = [\min_{\eta=1}^t(\text{Bel}_\eta), \min_{\eta=1}^t(\text{Pl}_\eta)]$, then $\tilde{h}_\eta^- \leq HFWPMSM_{DST}^\kappa(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) \leq \tilde{h}_\eta^+$.

The proofs of Theorem 3.5 and Theorem 3.6 are similar to those of Theorem 3.2 and Theorem 3.3, respectively. They are omitted here.

As follows, some transformations of the $HFWPMSM_{DST}$ operator are studied by setting different values of κ :

(1) When $\kappa = 1$, the $HFWPMSM_{DST}$ operator is transformed into the hesitant fuzzy weighted PA operator, that is

$$HFWPMSM_{DST}^1(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) = \left[\frac{1}{t} \sum_{\eta=1}^t \bar{\theta}_\eta \text{Bel}_\eta, \frac{1}{t} \sum_{\eta=1}^t \bar{\theta}_\eta \text{Pl}_\eta \right]. \tag{53}$$

(2) When $\kappa = 2$, it leads to the following expression:

$$\begin{aligned}
& HFWPMSM_{DST}^2(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) \tag{54} \\
& = \left[\frac{t}{C_t^2} \left(\sum_{1 \leq \eta_1 < \eta_2 \leq t} \bar{\theta}_{\eta_1} \bar{\theta}_{\eta_2} \text{Bel}_{\eta_1} \text{Bel}_{\eta_2} \right)^{\frac{1}{2}}, \frac{t}{C_t^2} \left(\sum_{1 \leq \eta_1 < \eta_2 \leq t} \bar{\theta}_{\eta_1} \bar{\theta}_{\eta_2} \text{Pl}_{\eta_1} \text{Pl}_{\eta_2} \right)^{\frac{1}{2}} \right].
\end{aligned}$$

(3) When $\kappa = t$, it leads to the following expression:

$$HFWPMSM_{DST}^t(\tilde{h}_1, \tilde{h}_2, \dots, \tilde{h}_t) = \left[t \left(\prod_{j=1}^t \bar{\theta}_{\eta_j} \text{Bel}_{\eta_j} \right)^{\frac{1}{t}}, t \left(\prod_{j=1}^t \bar{\theta}_{\eta_j} \text{Pl}_{\eta_j} \right)^{\frac{1}{t}} \right]. \tag{55}$$

The $HFWPMSM_{DST}$ operator can only deal with the MCDM problems where the weights of criteria are crisp numbers. However, in practical MCDM problems, decision makers could define the weights by HFNs. Therefore, it is important to define a novel $HFWPMSM_{DST}$ operator that can consider the weights of criteria represented by HFNs.

Definition 3.3. Let $\tilde{H} = \{\tilde{h}_\eta | \tilde{h}_\eta = [\text{Bel}_\eta, \text{Pl}_\eta], \eta = 1, 2, \dots, t\}$ be the corresponding BI set of HFS $H = \{h_\eta | h_\eta, \eta = 1, 2, \dots, t\}$. \bar{w}_η denote the weight of h_η in the form of HFN. $\bar{w}_\eta = [\bar{\text{Bel}}_\eta, \bar{\text{Pl}}_\eta]$ denotes the corresponding BI of \bar{w}_η . $\bar{w} = [\bar{w}_1, \bar{w}_2, \dots, \bar{w}_t]$ is a normalised interval weight vector (NIWV) when it meets the following two necessary

conditions (Wang et al. 2006; Liu et al. 2019a; Liu et al. 2020e):

(1) There exists at least a NIWV $a = (a_1, a_2, \dots, a_t) \in A$.

where $A = \{a = (a_1, a_2, \dots, a_t) | \overline{Bel}_\eta \leq a_\eta \leq \overline{Pl}_\eta, \eta = 1, 2, \dots, t, \sum_{\eta=1}^t a_\eta = 1\}$ is a set of NIWVs;

(2) \overline{Bel}_η and \overline{Pl}_η , ($\eta = 1, 2, \dots, t$) are all attainable in A. The definition of the $HF\overline{W}PMSM_{DST}$ operator of $\widetilde{h}_1, \widetilde{h}_2, \dots$, and \widetilde{h}_t is as follows:

$$HF\overline{W}PMSM_{DST}^{(\kappa)}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) = \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k \frac{t\overline{\omega}_{\eta_j} (1 + T(\widetilde{h}_{\eta_j})) \widetilde{h}_{\eta_j}}{\sum_{\eta=1}^t \overline{\omega}_\eta (1 + T(\widetilde{h}_\eta))}}{C_t^\kappa} \right)^{\frac{1}{\kappa}}, \quad (56)$$

where $T(\widetilde{h}_\eta) = \sum_{\tau=1, \tau \neq \eta}^t Sup(\widetilde{h}_\eta, \widetilde{h}_\tau)$, $Sup(\widetilde{h}_\eta, \widetilde{h}_\tau) = 1 - d(\widetilde{h}_\eta, \widetilde{h}_\tau)$, and $Sup(\widetilde{h}_\eta, \widetilde{h}_\tau)$ denotes the support degree for \widetilde{h}_η from \widetilde{h}_τ .

Theorem 3.7. Let $\widetilde{H} = \{\widetilde{h}_\eta | \widetilde{h}_\eta = [Bel_\eta, Pl_\eta], \eta = 1, 2, \dots, t\}$ be the corresponding BI set of HFS $H = \{h_\eta | h_\eta, \eta = 1, 2, \dots, t\}$. $W = (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_t)$ with $\overline{w}_\eta = [\overline{Bel}_\eta, \overline{Pl}_\eta]$ for $\eta = 1, 2, \dots, t$ is a NIWV. The aggregation result calculated by the $HF\overline{W}PMSM_{DST}$ operator is still a BI, which can be expressed by:

$$HF\overline{W}PMSM_{DST}^{(\kappa)}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) = \left[\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k \frac{t\overline{Bel}_{\eta_j} (1 + T(\widetilde{h}_{\eta_j})) Bel_{\eta_j}}{\sum_{\eta=1}^t \overline{Bel}_\eta (1 + T(\widetilde{h}_\eta))}}{(C_t^\kappa)^2} \right)^{\frac{1}{\kappa}}, \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k \frac{t\overline{Pl}_{\eta_j} (1 + T(\widetilde{h}_{\eta_j})) Pl_{\eta_j}}{\sum_{\eta=1}^t \overline{Pl}_\eta (1 + T(\widetilde{h}_\eta))}}{(C_t^\kappa)^2} \right)^{\frac{1}{\kappa}} \right]. \quad (57)$$

To simplify the above operator, let $\overline{\theta}_\eta = \frac{\overline{\omega}_\eta (1 + T(\widetilde{h}_\eta))}{\sum_{\eta=1}^t \overline{\omega}_\eta (1 + T(\widetilde{h}_\eta))}$, then Eq. (57) is given as follows:

$$HF\overline{W}PMSM_{DST}^{(\kappa)}(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) = \left[\left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\overline{\theta}_{\eta_j} \overline{Bel}_{\eta_j}}{(C_t^\kappa)^2} \right)^{\frac{1}{\kappa}}, \left(\frac{\sum_{1 \leq \eta_1 < \dots < \eta_k \leq t} \prod_{j=1}^k t\overline{\theta}_{\eta_j} Pl_{\eta_j}}{(C_t^\kappa)^2} \right)^{\frac{1}{\kappa}} \right]. \quad (58)$$

Theorem 3.8 (Commutativity). Let $\widetilde{h}_\eta = [Bel_\eta, Pl_\eta]$ ($\eta = 1, 2, \dots, t$) be the corresponding BIs of HFNs h_η ($\eta = 1, 2, \dots, t$). $W = (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_t)$ with $\overline{w}_\eta = [\overline{Bel}_\eta, \overline{Pl}_\eta]$ for $\eta = 1, 2, \dots, t$ is a normalised interval weight vector.

If $(\widetilde{h}'_1, \widetilde{h}'_2, \dots, \widetilde{h}'_t)$ is any permutation of $(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t)$, then $HF\overline{W}PMSM_{DST}^\kappa(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) = HF\overline{W}PMSM_{DST}^\kappa(\widetilde{h}'_1, \widetilde{h}'_2, \dots, \widetilde{h}'_t)$.

Theorem 3.9 (Boundedness). Let $\widetilde{h}_\eta = [Bel_\eta, Pl_\eta]$ ($\eta = 1, 2, \dots, t$) be the corresponding BIs of HFNs h_η ($\eta = 1, 2, \dots, t$). $W = (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_t)$ with $\overline{w}_\eta = [\overline{Bel}_\eta, \overline{Pl}_\eta]$ for $\eta = 1, 2, \dots, t$ is a normalised interval weight vector. If $\widetilde{h}_\eta^+ = [\max_{\eta=1}^t (Bel_\eta), \max_{\eta=1}^t (Pl_\eta)]$ and $\widetilde{h}_\eta^- = [\min_{\eta=1}^t (Bel_\eta), \min_{\eta=1}^t (Pl_\eta)]$, then $\widetilde{h}_\eta^- \leq HF\overline{W}PMSM_{DST}^\kappa(\widetilde{h}_1, \widetilde{h}_2, \dots, \widetilde{h}_t) \leq \widetilde{h}_\eta^+$.

The proofs of Theorem 3.8 and Theorem 3.9 are similar to those of Theorem 3.2 and Theorem 3.3, respectively. They are omitted here.

4 New MCDM method with HFNs in the framework of DST

In this section, based on the proposed $HFWPMSM_{DST}$ operator and $HF\bar{W}PMSM_{DST}$ operator, a novel MCDM method is proposed to solve MCDM problems with HFNs in the framework of DST.

Assume there is an MCDM problem with m alternatives, denoted by $T = \{t_1, t_2, \dots, t_m\}$ and η criteria, denoted by $R = \{r_1, r_2, \dots, r_\eta\}$. If the weights of criteria are real numbers, then the weight vector can be represented by $w = (w_1, w_2, \dots, w_\eta)$, satisfying $w_i > 0$ and $\sum_{i=1}^{\eta} w_i = 1$ ($i = 1, 2, \dots, \eta$), or if the weights of criteria are HFNs, $\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_\eta)$, such that $\bar{w}_l = [Bel_l, Pl_l]$ ($l = 1, 2, \dots, \eta$) is a NIWV. Let $B = (b_{ij})_{m \times \eta}$ be a decision matrix, where b_{ij} is an assessed value in the form of HFN, i.e. the evaluation value of the i -th alternative with respect to the j -th local criterion ($b_{ij}, i = 1$ to $m; j = 1$ to η). To select the optimal alternative is the ultimate goal of the MCDM problem.

The detailed steps of using the MCDM method to solve an MCDM problem are as follows:

Step 1. Normalise the decision matrix $B = (b_{ij})_{m \times \eta}$.

Step 2. Convert HFN b_{ij} to BI \tilde{h}_{ij} .

Step 3. Calculate $Sup(\tilde{h}_{ij}, \tilde{h}_{i\tau})(i = 1, 2, \dots, m; j, \tau = 1, 2, \dots, \eta; j \neq \tau)$, that is $Sup(\tilde{h}_{ij}, \tilde{h}_{i\tau}) = 1 - d(\tilde{h}_{ij}, \tilde{h}_{i\tau})(i = 1, 2, \dots, m; j, \tau = 1, 2, \dots, \eta; j \neq \tau)$, where $d(\tilde{h}_{ij}, \tilde{h}_{i\tau})$ can be calculated by the Jusselme distance in the framework of DST.

Step 4. Calculate $T(\tilde{h}_{ij})$ of \tilde{h}_{ij} by the other $\tilde{h}_{i\tau}$, that is $T(\tilde{h}_{ij}) = \sum_{\tau=1, j \neq \tau}^{\eta} Sup(\tilde{h}_{ij}, \tilde{h}_{i\tau})(i = 1, 2, \dots, m; j, \tau = 1, 2, \dots, \eta; j \neq \tau)$.

Step 5. Calculate $\bar{\theta}_i$ and $\bar{\theta}_l$. If $\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_\eta)$ with $\bar{w}_l = [Bel_l, Pl_l]$ for $l = 1, 2, \dots, \eta$, satisfies $\sum_{i=1}^{\eta} Bel_i + max_j(Pl_j - Bel_j) \leq 1$ and $\sum_{i=1}^{\eta} Pl_i - max_j(Pl_j - Bel_j) \geq 1$, then \bar{w} is normalised. If $\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_\eta)$ is not a NIWV, then it can be converted into a NIWV by the following equations:

$$Bel_i' = \min_{\bar{w} \in [Bel_j, Pl_j], j=1, 2, \dots, \eta, \sum_{j=1}^{\eta} \bar{w}_j = 1} \bar{w}_i = \max \left\{ Bel_i, 1 - \sum_{j=1, j \neq i}^{\eta} Pl_j \right\}, i = 1, 2, \dots, \eta. \quad (59)$$

$$Pl_i' = \max_{\bar{w} \in [Bel_j, Pl_j], j=1, 2, \dots, \eta, \sum_{j=1}^{\eta} \bar{w}_j = 1} \bar{w}_i = \max \left\{ Pl_i, 1 - \sum_{j=1, j \neq i}^{\eta} Bel_j \right\}, i = 1, 2, \dots, \eta. \quad (60)$$

and then,

$$\bar{\theta}_{ij} = \frac{w_i(1 + T(\tilde{h}_{ij}))}{\sum_{\tau=1}^{\eta} \omega_{\tau} (1 + T(\tilde{h}_{i\tau}))} \quad \text{or} \quad \bar{\theta}_l = \frac{\bar{w}_l(1 + T(\tilde{h}_{l_j}))}{\sum_{\tau=1}^{\eta} \bar{\omega}_{\tau} (1 + T(\tilde{h}_{l_{\tau}}))}$$

Step 6. Obtain the comprehensive value \tilde{h}_i ($i = 1, 2, \dots, m$) of each alternative through the proposed $HFWPMSM_{DST}$ operator or $HF\bar{W}PMSM_{DST}$ operator.

Step 7. Calculate $SF_{DST}(\tilde{h}_i)$ and $AF_{DST}(\tilde{h}_i)$ by Eq. (35) and Eq. (36), respectively.

Step 8. Rank the alternatives to get the optimal alternative.

5 Illustration and demonstration

In this section, a practical decision making example is used to verify the applicability of the proposed method. Further, the section demonstrates the advantages of the method by comparing it with some existing methods.

5.1. Application example

Example 5.1 (Sevastjanov et al. 2015). The manager of a computer center at a university plans to select a new information system to improve work productivity. After preliminary screening, four alternatives $T = \{t_1, t_2, t_3, t_4\}$ remain on the candidate list, and four criteria are under consideration: the costs of the hardware software investment (r_1); the contribution

to the performance of the organisation (r_2); the effort to transfer from the current system (r_3); and the reliability of the outsourcing software developer (r_4). These criteria are denoted by $R = \{r_1, r_2, r_3, r_4\}$. The weight vector of the criteria $r_i (i = 1, 2, 3, 4)$ is $w = [0.3, 0.25, 0.25, 0.2]^T$. The assessment value $b_{ij} (i = 1, 2, 3, 4)$ is in the form of HFN, and the decision matrix for collection and processing is constructed, as shown in Table 1.

Table 1 The decision matrix represented by HFN

	r_1	r_2	r_3	r_4
t_1	{0.3}	{0.3,0.5}	{0.3,0.6,0.8}	{0.6}
t_2	{0.4,0.8}	{0.5}	{0.4,0.7,0.9}	{0.2}
t_3	{0.1,0.4}	{0.3,0.4}	{0.7,0.8}	{0.2,0.3,0.5}
t_4	{0.3,0.4}	{0.6}	{0.8,0.9}	{0.4}

Table 2 The BIs from the HFNs

	r_1	r_2	r_3	r_4
t_1	[0.3,0.3]	[0.3,0.5]	[0.3,0.8]	[0.6,0.6]
t_2	[0.4,0.8]	[0.5,0.5]	[0.4,0.9]	[0.2,0.2]
t_3	[0.1,0.4]	[0.3,0.4]	[0.7,0.8]	[0.2,0.5]
t_4	[0.3,0.4]	[0.6,0.6]	[0.8,0.9]	[0.4,0.4]

Table 3 The BPAs for $b_{ij} (i, j = 1, 2, 3, 4)$

	$S(T)$	$S(F)$	$S(T \text{ or } F)$		$S(T)$	$S(F)$	$S(T \text{ or } F)$
f_{11}	0.3	0.7	0	f_{31}	0.1	0.6	0.3
f_{12}	0.3	0.5	0.2	f_{32}	0.3	0.6	0.1
f_{13}	0.3	0.2	0.5	f_{33}	0.7	0.2	0.1
f_{14}	0.6	0.4	0	f_{34}	0.2	0.5	0.3
	$S(T)$	$S(F)$	$S(T \text{ or } F)$		$S(T)$	$S(F)$	$S(T \text{ or } F)$
f_{21}	0.4	0.2	0.4	f_{41}	0.3	0.6	0.1
f_{22}	0.5	0.5	0	f_{42}	0.6	0.4	0
f_{23}	0.4	0.1	0.5	f_{43}	0.8	0.1	0.1
f_{24}	0.2	0.8	0	f_{44}	0.4	0.6	0

The detailed steps of using the MCDM method based on $HFWPMSM_{DST}$ operator to solve the information system selection problem are as follows:

Step 1: Normalise the decision matrix $B = (b_{ij})_{m \times n}$. Because each criterion is beneficial, normalisation is not required.

Step 2: Convert HFN b_{ij} to BI \tilde{h}_{ij} and get BPAs. Table 2 lists the BIs for $b_{ij} (i, j = 1, 2, 3, 4)$ and Table 3 lists the BPAs for $b_{ij} (i, j = 1, 2, 3, 4)$.

Step 3: Calculate $Sup(\tilde{h}_{ij}, \tilde{h}_{i\tau}) (i = 1, 2, 3, 4; j, \tau = 1, 2, 3, 4; j \neq \tau)$, then

$$Sup(\tilde{h}_{1j}, \tilde{h}_{1\tau}) = Sup(\tilde{h}_{1j}, \tilde{h}_{1\tau}) = \begin{bmatrix} 1.0000 & 0.8586 & 0.6464 & 0.7000 \\ 0.8586 & 1.0000 & 0.7879 & 0.7764 \\ 0.6464 & 0.7879 & 1.0000 & 0.7450 \\ 0.7000 & 0.7764 & 0.7450 & 1.0000 \end{bmatrix},$$

$$Sup(\tilde{h}_{2j}, \tilde{h}_{2\tau}) = Sup(\tilde{h}_{2j}, \tilde{h}_{2\tau}) = \begin{bmatrix} 1.0000 & 0.7764 & 0.9293 & 0.5528 \\ 0.7764 & 1.0000 & 0.7085 & 0.7000 \\ 0.9293 & 0.7085 & 1.0000 & 0.4852 \\ 0.5528 & 0.7000 & 0.4852 & 1.0000 \end{bmatrix},$$

$$Sup(\widetilde{h}_3, \widetilde{h}_{3\tau}) = Sup(\widetilde{h}_{3\tau}, \widetilde{h}_3) = \begin{bmatrix} 1.0000 & 0.8586 & 0.4901 & 0.9000 \\ 0.8586 & 1.0000 & 0.6000 & 0.9000 \\ 0.4901 & 0.6000 & 1.0000 & 0.5877 \\ 0.9000 & 0.9000 & 0.5877 & 1.0000 \end{bmatrix},$$

$$Sup(\widetilde{h}_4, \widetilde{h}_{4\tau}) = Sup(\widetilde{h}_{4\tau}, \widetilde{h}_4) = \begin{bmatrix} 1.0000 & 0.7450 & 0.5000 & 0.9293 \\ 0.7450 & 1.0000 & 0.7450 & 0.8000 \\ 0.5000 & 0.7450 & 1.0000 & 0.5472 \\ 0.9293 & 0.8000 & 0.5472 & 1.0000 \end{bmatrix}.$$

Step 4: Calculate $T(\widetilde{h}_{ij})$ of \widetilde{h}_{ij} by $\widetilde{h}_{i\tau}$, that is,

$$T(\widetilde{h}_{ij}) = \begin{bmatrix} 2.2050 & 2.4228 & 2.1794 & 2.2214 \\ 2.2585 & 2.1848 & 2.1230 & 1.7380 \\ 2.2487 & 2.3586 & 1.6778 & 2.3877 \\ 2.1743 & 2.2901 & 1.7923 & 2.2765 \end{bmatrix}.$$

Step 5: Calculate the $\overline{\theta}_{ij}$, and obtain

$$\overline{\theta}_{ij} = \begin{bmatrix} 0.2953 & 0.2628 & 0.2441 & 0.1979 \\ 0.3151 & 0.2567 & 0.2517 & 0.1765 \\ 0.3083 & 0.2656 & 0.2118 & 0.2143 \\ 0.3044 & 0.2629 & 0.2232 & 0.2095 \end{bmatrix}.$$

Step 6: Obtain the comprehensive value \widetilde{h}_i ($i = 1, 2, \dots, m$) of each alternative using the proposed $HFWPMSM_{DST}$ operator shown in Equation (51) ($\kappa = 2$).

$$HFWPMSM_{DST}(t_1) = [0.3571, 0.5263],$$

$$HFWPMSM_{DST}(t_2) = [0.3806, 0.6109],$$

$$HFWPMSM_{DST}(t_3) = [0.2825, 0.5027],$$

$$HFWPMSM_{DST}(t_4) = [0.5025, 0.5552].$$

Step 7: Calculate $SF_{DST}(t_i)$ by Eq. (35), and obtain $SF_{DST}(t_1) = 0.4417$, $SF_{DST}(t_2) = 0.4958$, $SF_{DST}(t_3) = 0.3926$, $SF_{DST}(t_4) = 0.5289$.

Step 8: Rank the alternatives and get the best alternative. Given that $SF_{DST}(t_4) > SF_{DST}(t_2) > SF_{DST}(t_1) > SF_{DST}(t_3)$, the ranking order is $t_4 > t_2 > t_1 > t_3$, and the best information system is t_4 .

5.2. Effect of the parameters κ on ranking results

In order to reflect the influence with the different value of the parameter κ on the ranking results, we use a different parameter κ in the proposed new method to rank the above example. Table 4 lists the ranking results, and Figure 1 describes the change curve of $SF_{DST}(t_i)$ based on the value of κ ($\kappa = 1, 2, 3, 4$).

Table 4 Ranking results of the alternatives with respect to different parameters.

κ	$SF_{DST}(t_i)$	Ranking Orders
$\kappa = 1$	$\widetilde{S}_1 = 0.4467, \widetilde{S}_2 = 0.5163, \widetilde{S}_3 = 0.4039, \widetilde{S}_4 = 0.5378$	$t_4 > t_2 > t_1 > t_3$
$\kappa = 2$	$\widetilde{S}_1 = 0.4417, \widetilde{S}_2 = 0.4958, \widetilde{S}_3 = 0.3926, \widetilde{S}_4 = 0.5289$	$t_4 > t_2 > t_1 > t_3$
$\kappa = 3$	$\widetilde{S}_1 = 0.4372, \widetilde{S}_2 = 0.4683, \widetilde{S}_3 = 0.3826, \widetilde{S}_4 = 0.5197$	$t_4 > t_2 > t_1 > t_3$
$\kappa = 4$	$\widetilde{S}_1 = 0.4329, \widetilde{S}_2 = 0.4278, \widetilde{S}_3 = 0.3741, \widetilde{S}_4 = 0.5105$	$t_4 > t_1 > t_2 > t_3$

Note: \widetilde{S}_i is abbreviation of score value $SF_{DST}(t_i)$

It can be seen from Table 4 that the ranking results are slightly different as the value of κ changes. However, the optimal information system is still t_4 , which does not change with the change of the parameter κ . This is probably because the proposed method allows more interaction criteria to increase the value of κ . The presented new method does not take into consideration the interrelationships between criteria when $\kappa = 1$, and the ranking result is essentially different from the results when $\kappa = 2$, $\kappa = 3$, $\kappa = 4$. Obviously, this illustrates the importance of considering the interrelationships between criteria because there is a general interrelationship between two or more criteria in practical MCDM problems.

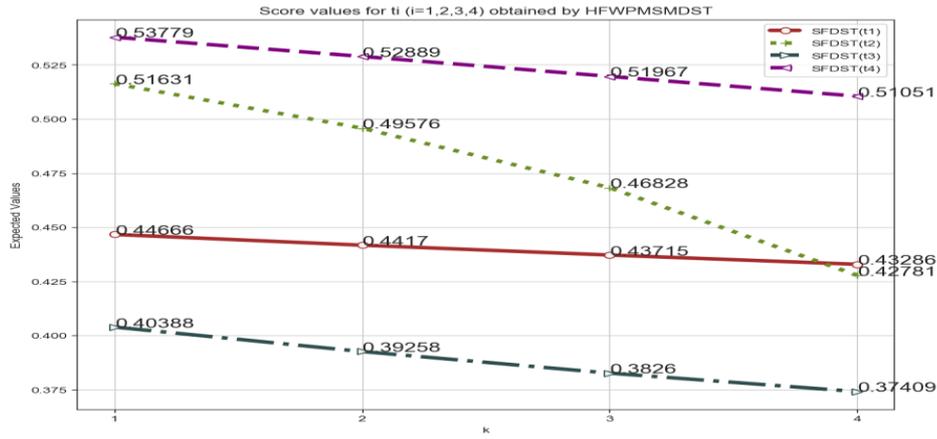


Fig 1. $SF_{DST}(t_i)$ ($i = 1,2,3,4$) based on the $HFWPMSM_{DST}$ operator.

In addition, from Figure 1, we also can find that the score values obtained by the $HFWPMSM_{DST}$ operator become smaller as the parameter κ increases for the same alternative. In practical MCDM problems, decision makers can choose the appropriate value in accordance with their risk preferences. In general, $\kappa = \lceil \eta/2 \rceil$ is a suitable value, where symbol $\lceil \cdot \rceil$ means round function and η is the number of criteria.

5.3. Verification of the effectiveness

In order to prove that the proposed method based on the $HFWPMSM_{DST}$ operator is reasonable and effective, Example 5.1 can be solved by using six existing methods: Xia and Xu (2011)'s methods, based on the generalised hesitant fuzzy weighted averaging (GHFWA) operator and the generalised hesitant fuzzy weighted geometric (GHFWG) operator, Zhang (2013b)'s method, based on the hesitant fuzzy weighted power average (HFWPA) operator, Qin and Liu (2015)'s method, based on the weighted hesitant fuzzy Maclaurin symmetric mean ($WHFMSM$) operator, Sevastjanov and Dymova (2015)'s methods, based on the hesitant weighted arithmetical mean in the framework of DST ($HWAM_{DST}$) operator and the hesitant weighted geometric mean in the framework of DST ($HWGM_{DST}$) operator, where we let $k = 2$ for the proposed new method and Qin and Liu (2015)'s method; let $\lambda = 1$ for Xia and Xu (2011)'s method. Table 5 lists the ranking results of different methods.

Table 5 Ranking results of different methods for Example 5.1

Methods	Score Values	Ranking Order
$GHFWA_1$ (Xia and Xu 2011)	$S_1 = 0.4793, S_2 = 0.5685,$ $S_3 = 0.4692, S_4 = 0.6118$	$t_4 > t_2 > t_1 > t_3$
$GHFWG_1$ (Xia and Xu 2011)	$S_1 = 0.4254, S_2 = 0.4621,$ $S_3 = 0.3573, S_4 = 0.5122$	$t_4 > t_2 > t_1 > t_3$
$HFWPA$ (Zhang 2013b)	$S_1 = 0.4727, S_2 = 0.5465,$ $S_3 = 0.4360, S_4 = 0.6056$	$t_4 > t_2 > t_1 > t_3$
$WHFMSM^2$ (Qin et al. 2015)	$S_1 = 0.4689, S_2 = 0.4828,$ $S_3 = 0.102, S_4 = 0.5492$	$t_4 > t_2 > t_1 > t_3$
$HWAM_{DST}$ (Sevastjanov et al. 2015)	$\tilde{S}_1 = 0.1119, \tilde{S}_2 = 0.1269,$ $\tilde{S}_3 = 0.1050, \tilde{S}_4 = 0.1369$	$t_4 > t_2 > t_1 > t_3$
$HWGM_{DST}$ ((Sevastjanov et al. 2015))	$\tilde{S}_1 = 0.4225, \tilde{S}_2 = 0.4617,$ $\tilde{S}_3 = 0.3716, \tilde{S}_4 = 0.5125$	$t_4 > t_2 > t_1 > t_3$
$HFWPMSM_{DST}^2$ in this paper	$\tilde{S}_1 = 0.4417, \tilde{S}_2 = 0.4958,$ $\tilde{S}_3 = 0.3926, \tilde{S}_4 = 0.5289$	$t_4 > t_2 > t_1 > t_3$

Notes: S_i is the abbreviation of score value $SF(t_i)$; \tilde{S}_i is the abbreviation of score value $SF_{DST}(t_i)$

In Table 5, the ranking result of the proposed method based on the $HFWPMSM_{DST}$ operator is the same as those introduced in (Zhang 2013b; Qin et al. 2015; Sevastjanov et al. 2015; Xia and Xu 2011), respectively. Synchronously, it is possible to apply the proposed new method to deal with the examples in (Qin et al. 2015; Xia and Xu 2011).

Example 5.2 (Qin et al. 2015). A journal editor hopes to select the best paper from a list of papers to improve the quality of journal papers. Firstly, we give the necessary information (condition) in the process of evaluation. After pre-evaluation, five papers t_i ($i = 1, 2, \dots, 5$) have been designated for further evaluation. The peer reviewers must make a decision according to the following four criteria: the originality, novelty, and significance (r_1), the technical quality of work (r_2), the comprehensibility and presentation of paper (r_3), the overall impression (r_4). The criteria weight vector is given as $w = [0.3, 0.2, 0.2, 0.3]^T$. The five papers t_i ($i = 1, 2, \dots, 5$) were evaluated in anonymity with hesitant fuzzy information by experts under the above four criteria r_j ($j = 1, 2, 3, 4$), as listed in Table 6, and Table 7 lists the ranking results of different methods, where we assume that $\kappa = 2$ for the proposed new method and Qin and Liu (2015)'s method.

Table 6 The decision matrix represented by HFN (Qin et al. 2015) in Example 5.2.

	r_1	r_2	r_3	r_4
t_1	{0.6,0.8}	{0.2,0.6,0.8}	{0.6}	{0.4,0.5}
t_2	{0.4,0.7,0.9}	{0.2,0.4}	{0.6,0.9}	{0.5}
t_3	{0.5}	{0.7,0.8}	{0.3,0.5,0.7}	{0.5,0.7}
t_4	{0.4,0.5,0.6}	{0.1,0.3}	{0.4,0.9}	{0.3}
t_5	{0.4,0.7}	{0.2,0.3}	{0.8}	{0.3,0.4,0.8}

Table 7 Ranking results of different methods for Example 5.2.

Methods	Score Values	Ranking Orders
$HFMSM^2$ (Qin et al. 2015)	$S_1 = 0.5462, S_2 = 0.5071,$ $S_3 = 0.5731, S_4 = 0.3562, S_5 = 0.4690$	$t_3 > t_1 > t_2 > t_5 > t_4$
$HFWPMSM_{DST}^2$ in this paper	$S_1 = 0.5365, S_2 = 0.5250,$ $S_3 = 0.5551, S_4 = 0.3810, S_5 = 0.5071$	$t_3 > t_1 > t_2 > t_5 > t_4$

Notes: S_i is the abbreviation of score value $SF(t_i)$; \tilde{S}_i is the abbreviation of score value $SF_{DST}(t_i)$

From Table 7, the ranking results of all the methods are obviously the same, i.e., $t_3 > t_1 > t_2 > t_5 > t_4$.

Example 5.3 (Xia and Xu 2011). The enterprise's board of directors is to plan the development of large projects (strategy initiatives) for the following five years. There are four possible projects t_i ($i = 1, 2, 3, 4$) to be evaluated, and the definitions of the four criteria are as follows: financial perspective (r_1); customer satisfaction (r_2); internal business process perspective (r_3); learning and growth perspective (r_4). The criteria weight vector is given as $w = [0.2, 0.3, 0.15, 0.35]^T$. The assessment value b_{ij} of criteria r_j ($j = 1, 2, 3, 4$) with alternative t_i ($i = 1, 2, 3, 4$) is in the form of HFN, and Table 8 lists the decision matrix. Table 9 lists the ranking results of different methods, where we assume that $\kappa = 2$ for the proposed new method and $\lambda = 1$ for Xia and Xu (2011)'s method.

Table 8 The decision matrix represented by HFNs in Example 5.3.

	r_1	r_2	r_3	r_4
t_1	{0.2,0.4,0.7}	{0.2,0.6,0.8}	{0.2,0.3,0.6,0.7,0.9}	{0.3,0.4,0.5,0.7,0.8}
t_2	{0.2,0.4,0.7,0.9}	{0.1,0.2,0.4,0.5}	{0.3,0.4,0.6,0.9}	{0.5,0.6,0.8,0.9}
t_3	{0.3,0.5,0.6,0.7}	{0.2,0.4,0.5,0.6}	{0.3,0.5,0.7,0.8}	{0.2,0.5,0.6,0.7}
t_4	{0.3,0.5,0.6}	{0.2,0.4}	{0.5,0.6,0.7}	{0.8,0.9}

Table 9 Ranking results of different methods for Example 5.3.

Methods	Score Values	Ranking Orders
$GHFWA_1$ (Xia and Xu 2011)	$S_1 = 0.5612, S_2 = 0.6009,$ $S_3 = 0.5178, S_4 = 0.6524$	$t_4 > t_2 > t_1 > t_3$
$GHFWG_1$ (Xia and Xu 2011)	$S_1 = 0.4744, S_2 = 0.4625,$ $S_3 = 0.4661, S_4 = 0.5130$	$t_4 > t_1 > t_3 > t_2$
$HFWPMSM_{DST}^2$ in this paper	$S_1 = 0.5032, S_2 = 0.5157,$ $S_3 = 0.4549, S_4 = 0.5273$	$t_4 > t_2 > t_1 > t_3$

Notes: S_i is the abbreviation of score value $SF(t_i)$; \tilde{S}_i is the abbreviation of score value $SF_{DST}(t_i)$

From Table 9, the ranking results of all the methods for the alternatives are obviously the same, i.e., $t_4 > t_2 > t_1 > t_3$. Through the verification of examples 5.1, 5.2 and 5.3, it is clear that the proposed method is effective and reasonable.

5.4. The advantages compared with the existing methods

The reasonability and effectiveness of the presented method have been verified. However, since the ranking results obtained in the previous section are the same, it is difficult to directly observe the advantages of the method proposed in this paper and the disadvantages of the methods in (Zhang 2013b; Qin et al. 2015). Accordingly, the presented method and the existing two methods can be applied to process three novel application examples.

5.4.1. Considering the interrelationships among criteria

Example 5.4 (Gao et al. 2019). A company wants to rent out dormitories to its employees to build a stable talent team. Now, there are four alternatives t_i ($i = 1,2,3,4$) from four communities, and four criteria are defined as follows: comfortability (r_1), convenience (r_2), consisting of cost (r_3), and living spaces (r_4). The criteria weight vector is given as $w = [0.15, 0.25, 0.3, 0.3]^T$. The assessment value b_{ij} of criterion r_j ($1,2,3,4$) with alternative t_i ($i = 1,2,3,4$) is in the form of HFN, and Table 10 lists the decision matrix. Table 11 lists the ranking results of different methods, where we let $k = 1,2$ for the proposed new method and let $k = 2$ for Qin and Liu (2015)'s method.

Table 10 The decision matrix represented by HFNs in Example 5.4.

	r_1	r_2	r_3	r_4
t_1	{0.5,0.6}	{0.2,0.3,0.5}	{0.3,0.4,0.6,0.7}	{0.4,0.5}
t_2	{0.2}	{0.3,0.5,0.7}	{0.5}	{0.3,0.7}
t_3	{0.6,0.7}	{0.5,0.6,0.8}	{0.3,0.4}	{0.3,0.5,0.6,0.7}
t_4	{0.4,0.6}	{0.3,0.4,0.6,0.7}	{0.5}	{0.3,0.5,0.6}

From Table 11, the ranking results based on $HFWPMSM_{DST}^1$ operator are obviously the same as those base on the HFPA operators in (Zhang 2013b), $GHFWA_1$ and $GHFWG_1$ operators in (Xia and Xu 2011). The methods based on $HFWPMSM_{DST}^1$ operator, HFPA operator in (Zhang 2013b), $GHFWA_1$ and $GHFWG_1$ operators in (Xia and Xu 2011) do not take into consideration the interrelationships between criteria. Obviously, the ranking result is not reasonable, because there are certain correlations between the criteria in this example. i.e., correlation between comfortability (r_1) and convenience (r_2), correlation between consisting of cost (r_3) and living spaces (r_4). Generally, the enlargement of living space will improve comfort, convenience and consisting of cost. Therefore, it is important to take into consideration the interrelationships between them. Synchronously, it is also not difficult to observe that the ranking result based on the $HFWPMSM_{DST}^2$ operator is identical to that based on the $WHFMSM^2$ operator in (Qin et al. 2015). Obviously, the method based on the $HFWPMSM_{DST}^2$ operator takes into consideration the interrelationships between criteria, and it is similar to the method based on $WHFMSM^2$ operator in (Qin et al. 2015). Evidently, it is more reasonable to fully consider the interrelationships in actual selection of talent apartment.

Table 11 Ranking results of different methods for Example 5.4

Methods	Score Values	Ranking Orders
$GHFWA_1$ (Xia and Xu 2011)	$S_1 = 0.4646, S_2 = 0.4787,$ $S_3 = 0.5442, S_4 = 0.4994$	$t_3 > t_4 > t_2 > t_1$
$GHFWG_1$ (Xia and Xu 2011)	$S_1 = 0.4226, S_2 = 0.4234,$ $S_3 = 0.4952, S_4 = 0.4796$	$t_3 > t_4 > t_2 > t_1$
$HFWPA$ (Zhang 2013b)	$S_1 = 0.4464, S_2 = 0.4566,$ $S_3 = 0.5269, S_4 = 0.4801$	$t_3 > t_4 > t_2 > t_1$
$WHFMSM^2$ (Qin et al. 2015)	$S_1 = 0.4585, S_2 = 0.4086,$ $S_3 = 0.5471, S_4 = 0.4911$	$t_3 > t_4 > t_1 > t_2$
$HFWPMSM_{DST}^1$ in this paper	$\tilde{S}_1 = 0.4553, \tilde{S}_2 = 0.4597,$ $\tilde{S}_3 = 0.5185, \tilde{S}_4 = 0.4847$	$t_3 > t_4 > t_2 > t_1$
$HFWPMSM_{DST}^2$ in this paper	$\tilde{S}_1 = 0.4480, \tilde{S}_2 = 0.4402,$ $\tilde{S}_3 = 0.5123, \tilde{S}_4 = 0.4782$	$t_3 > t_4 > t_1 > t_2$

Notes: S_i is the abbreviation of score value $SF(t_i)$; \tilde{S}_i is the abbreviation of score value $SF_{DST}(t_i)$

5.4.2. Reducing the influence of extreme evaluation values

Example 5.5. In a real decision-making process, due to the preferences of experts, some unreasonable or extreme evaluation values may be provided, such as too high or too low values. Thus, the real ranking results may be affected. To illustrate this case, based on Example 5.3, the value of b_{11} is changed to $\{0.001, 0.4, 0.7\}$ and the value of b_{21} is changed to $\{0.2, 0.4, 0.9, 0.999\}$. Subsequently Table 12 lists the decision matrix, and Table 13 lists the ranking results of different methods.

Table 12 The decision matrix represented by HFNs in Example 5.5.

	r_1	r_2	r_3	r_4
t_1	$\{0.001, 0.4, 0.7\}$	$\{0.2, 0.6, 0.8\}$	$\{0.2, 0.3, 0.6, 0.7, 0.9\}$	$\{0.3, 0.4, 0.5, 0.7, 0.8\}$
t_2	$\{0.2, 0.4, 0.7, 0.9999\}$	$\{0.1, 0.2, 0.4, 0.5\}$	$\{0.3, 0.4, 0.6, 0.9\}$	$\{0.5, 0.6, 0.8, 0.9\}$
t_3	$\{0.3, 0.5, 0.6, 0.7\}$	$\{0.2, 0.4, 0.5, 0.6\}$	$\{0.3, 0.5, 0.7, 0.8\}$	$\{0.2, 0.5, 0.6, 0.7\}$
t_4	$\{0.3, 0.5, 0.6\}$	$\{0.2, 0.4\}$	$\{0.5, 0.6, 0.7\}$	$\{0.8, 0.9\}$

Table 13 Ranking results of different methods for Example 5.5.

Methods	Score Values	Ranking Order
$GHFWA_1$ (Xia and Xu 2011)	$S_1 = 0.5540, S_2 = 0.6584,$ $S_3 = 0.5178, S_4 = 0.6104$	$t_2 > t_4 > t_1 > t_3$
$GHFWG_1$ (Xia and Xu 2011)	$S_1 = 0.3841, S_2 = 0.4653,$ $S_3 = 0.4661, S_4 = 0.5130$	$t_4 > t_3 > t_2 > t_1$
$HFWPA$ (Zhang 2013b)	$S_1 = 0.4960, S_2 = 0.5529,$ $S_3 = 0.4613, S_4 = 0.5589$	$t_4 > t_2 > t_1 > t_3$
$WHFMSM^2$ (Qin et al. 2015)	$S_1 = 0.4876, S_2 = 0.5518,$ $S_3 = 0.5083, S_4 = 0.5514$	$t_2 > t_4 > t_3 > t_1$
$HFWPMSM_{DST}^2$ in this paper	$\tilde{S}_1 = 0.4786, \tilde{S}_2 = 0.5266,$ $\tilde{S}_3 = 0.4549, \tilde{S}_4 = 0.5273$	$t_4 > t_2 > t_1 > t_3$

Notes: S_i is the abbreviation of score value $SF(t_i)$; \tilde{S}_i is the abbreviation of score value $SF_{DST}(t_i)$

From Table 13, it is not difficult to observe that a high value $\{0.999\}$ and a low value $\{0.001\}$ have key effects on the

ranking results based on the $WHFMSM^2$ operator (Qin et al. 2015) and the $GHFWA_1$ and $GHFWG_1$ operators (Xia and Xu 2011). The ranking results based on the $WHFMSM^2$ and $GHFWA_1$ operators change directly from $t_4 > t_2 > t_3 > t_1$ to $t_2 > t_4 > t_3 > t_1$, and best alternative is changed from t_4 to t_2 . The ranking result based on the $GHFWG_1$ operator is also different from that obtained in Example 5.3. Obviously, the two ranking results mentioned above are unreasonable. Hence, in real decision making problems, a decision-maker can become a manipulator by giving some unreasonable values. However, the ranking results based on the $HFWPA$ operator (Zhang 2013b) and the proposed operators are still reasonable and the optimal large project is still t_4 . The reason is that the PA operator set smaller weights for these criteria by the support degrees between criteria to reduce the influence of extreme evaluation values.

5.4.3. The criterion weights denoted by HFNs

Example 5.6. It is easy to see that the ordinary operational laws of HFS could result in some unreasonable ranking results, because there exists some unfavorable properties and the criterion weights in the aggregation operators cannot be expressed by HFNs. Obviously, these are critical defects of ordinary operational laws for HFSs. In the framework of DST, when criteria and their weights are HFNs, the operational laws for HFSs proposed in this paper can overcome these critical defects and intermediate defuzzification processing does not need to be performed when solving the MCDM problems. To elucidate the advantage of the operators proposed in the paper, Example 5.1 was adjusted. the originally $w = [0.3, 0.25, 0.25, 0.2]^T$ in the example was transformed into $\hat{w} = (\{0.3, 0.4, 0.7\}, \{0.2, 0.4\}, \{0.1, 0.2, 0.5, 0.6\}, \{0.3, 0.5\})$. Specifically, the corresponding interval weights of \hat{w} are $\bar{w} = ([0.3, 0.7], [0.2, 0.4], [0.1, 0.6], [0.3, 0.5])$. According to Eqs. (59) and (60), it can be proved that \bar{w} is a NIWV. Table 14 lists the ranking results.

Table 14 Ranking results of different methods for Example 5.6.

Methods	Score Value	Ranking Order
$GHFWA_1$ (Xia and Xu 2011)	Cannot be counted	Cannot be ranked
$GHFWG_1$ (Xia and Xu 2011)	Cannot be counted	Cannot be ranked
$HFWPA$ (Zhang 2013b)	Cannot be counted	Cannot be ranked
$WHFMSM^2$ (Qin et al. 2015)	Cannot be counted	Cannot be ranked
$HFWPMSM_{DST}^2$ in this paper	$\tilde{S}_1 = 0.4505, \tilde{S}_2 = 0.4809,$ $\tilde{S}_3 = 0.3712, \tilde{S}_4 = 0.4987$	$t_4 > t_2 > t_1 > t_3$

Notes: S_i is the abbreviation of score value $SF(t_i)$; \tilde{S}_i is the abbreviation of score value $SF_{DST}(t_i)$

From Table 14, only the new method based on the $HF\bar{W}PMSM_{DST}$ operator can produce a ranking result, i.e., $t_4 > t_2 > t_1 > t_3$. Consequently, the operational laws for HFSs in the framework of DST can augment the function of aggregation operators.

Based on the above examples, the drawbacks of the $HFWPA$ operator, $WHFMSM$ operator, and $GHFWA_1$ and $GHFWG_1$ operators are concluded as follows:

- (1) The $HFWPA$ operator (Zhang 2013b) does not consider the interrelationships between criteria. In Example 5.4, we discussed that it is meaningful to take into account the interrelationships between criteria, but this operator can only handle the MCDM problems where all criteria are independent of each other. Further, since there are no variable parameters, the operator does not show the subject preference of the decision maker, and thus it is not suitable to capture the risk attitudes of some decision makers.
- (2) The $WHFMSM$ operator (Qin et al. 2015) only considers the interrelationships between criteria and cannot handle extreme evaluation values correctly.
- (3) The most obvious drawbacks of the $HFWPA$ (Zhang 2013b) operator and $WHFMSM$ (Qin et al. 2015) operator

are that they are not applicable when criteria weights are HFNs and some unfavorable ranking results could be obtained because of the limitations of the operational laws.

- (4) The $GHFWA_1$ and $GHFWG_1$ operators (Xia and Xu 2011) have all the limitations of the HFWPA and WHFMSM operators described above.

Table 15 Comparison results of the characteristics of different methods.

Methods	Whether eliminate the effects of biased values	Whether consider the interrelationships between criteria	Whether criteria weights can be denoted by HFNs	Whether it is free of the limitations of ordinary HFSs
$GHFWA_1$ (Xia and Xu 2011)	No	No	No	No
$GHFWG_1$ (Xia and Xu 2011)	No	No	No	No
$HFWPA$ (Zhang 2013b)	Yes	No	No	No
$WHFMSM^2$ (Qin et al. 2015)	No	Yes	No	No
$HFWPMSM_{DST}^2$ in this paper	Yes	Yes	Yes	Yes

In the following, we compare the properties of the proposed operators in this paper with the operators in (Xia and Xu 2011; Zhang 2013b; Qin et al. 2015). The comparison results are shown in Table 15. Obviously, the operators proposed in the paper can get rid of the drawbacks of the four existing operators and is more extensive and flexible in solving MCDM problems.

6 Conclusions

In this paper, an $HFPMSM_{DST}$ operator and an $HFWPMSM_{DST}$ operator are developed. The properties of these aggregation operators are discussed. Subsequently, a new MCDM method based on the $HFWPMSM_{DST}$ operator is proposed. A set of application examples are introduced to demonstrate the effectiveness and advantages of the proposed method. The demonstration results suggest that the method has the following characteristics at the same time: eliminate the effects of biased values; consider the interrelationships between criteria; deal with the situations where criteria weights are denoted by HFNs; overcome the limitations of traditional operational laws for HFSs. In the future, we will combine the developed aggregation operators with deep learning and apply them to recommendation system and text classification.

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Compliance with ethical standards

Conflict of interest The authors state that they have no conflict of interest.

Ethical standard This article does not contain any studies with human participants or animals performed by any of the

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