Axially compressed thin cylindrical shells: asymptotic limits for a nonlinear basic state

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Abstract

The work presented here revisits the classical bifurcation problem for an axially compressed thin elastic circular cylinder when the nonlinear pre-buckling bending deformations are taken into account. Suitable scaling arguments and singular perturbation methods enable us to explore the link with the more widely studied classical cases in which the uniform membrane solution was used for describing the basic state. Our novel theoretical results are confirmed by direct numerical simulations of the full boundary-value problems corresponding to the relevant buckling scenarios.

Keywords: cylindrical shells, boundary layers, nonlinear bending, perturbation methods, elastic stability.

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1 Introduction

A well-known simplification commonly introduced in the classical bifurcation analyses of thin elastic plates and shells is the use of linear membrane solutions for describing the corresponding pre-buckling deformations (e.g., [1]). This is a natural assumption for uniform rectangular thin plates compressed by in-plane forces acting in their midplane as it corresponds to the everyday experience: such an initially flat thin plate will continue to remain in its original state for as long as the magnitude of the compressive forces does not exceed a certain critical threshold (i.e., the buckling load). With regard to more complex geometries or loading scenarios, the situation is not quite so clear-cut because bending effects might be non-trivially involved prior to the onset of a bifurcation. A particular case in point is the edge-buckling of a stretched thin circular elastic plate under the action of a uniform transverse pressure. While bending deformations might seem to play a key role in the description of the basic state involved, numerical evidence [2, 3] indicates that nonlinear membrane (i.e., stretching) deformations prevail just before the bifurcation is trigerred. This observation was also confirmed by a couple of detailed asymptotic investigations [4, 5], in which the basic state was described by purely nonlinear membrane behaviour.

The general problem investigated in the following pages is related to the aforementioned matters. Broadly speaking, we are interested in the effect of *pre-buckling bending deformations* on the approximation of the critical loads associated with the buckling of axially compressed circular cylindrical shells. The classical theory involving a membrane basic state is well documented in the literature (e.g., [1, 6, 7]); the two volumes [8, 9] include a thorough discussion of experimental work as well. For further reference, we recall below the classical critical axial stress for an infinitely long, simply supported thin cylinder of radius R and thickness h (with $0 < h/R \ll 1$)

$$\sigma_{cr} \equiv \frac{E}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{R}\right) \simeq 0.605\left(\frac{Eh}{R}\right), \qquad (\nu = 0.3), \tag{1.1}$$

where E and ν denote, respectively, the Young's modulus and the Poisson's ratio of the cylinder.

Lack of agreement between experiments and theoretical predictions for circular cylinders encouraged the early investigators to consider the effect that the type of end restraints might have on the computed buckling loads. Kilcevsky [10] had explored analytically the case of a *free edge*, and discovered that the critical load for the radially symmetric modes obtained by taking into account the edge constraints was half of that stated in (1.1). A similar conclusion was reached independently by Hoff almost two decades later in a study [11] involving a semi-infinite thin elastic cylinder with a free edge, while Ohira [12] obtained a confirmation of the same result on the basis of a numerical solution of some related eigenvalue problems. In a follow-up work to [11], Nachbar & Hoff [13] relaxed the axial symmetry of the eigenmodes by employing the full Donnel-Mushtari-Vlasov (DMV) buckling equations; a refined set of (ad-hoc) equilibrium conditions was also derived for the deformed edge. As a result of this added generality, the critical buckling stress turned out to be 37% of the classical prediction (1.1), with the corresponding deformations being characterised by a regular circumferential "rippling" pattern that attenuated rapidly away from the free edge. Hoff & Rehfield [14] re-visited the axially compressed semi-infinite cylinder by considering a generalisation of the classical boundary conditions for a simply supported circular edge within the context of the DMV shell theory. By relaxing the in-plane end-restraints they proposed four different ways to constrain the circular edges. Two of the new boundary conditions were found to yield critical compressive stresses slightly over 50% of the classical value stated in (1.1), a result reinforced by a subsequent study that considered finite-length cylinders [15].

Hoff's work was largely based on DMV-type buckling equations with *constant* coefficients as his basic state was taken to be uniform. Such a restriction seems to have been largely dictated by his method of solution; this relied on closed-form expressions for the roots of the characteristic equation obtained upon substitution of the usual exponential trial solution into the differential equations. A particularly simple form of the characteristic roots was originally derived by Nachbar [16] and required special re-scaling of the DMV buckling equations (which is not applicable in the case of a non-uniform basic state). Since closed-form solutions of those equations are therefore available, one can actually write down the relevant determinantal equation for identifying the critical loads. In the aforementioned works, Hoff and his associates simplified their determinantal equations, which eventually permitted them to obtain compact formulae for the corresponding buckling loads. A direct numerical strategy was also pursued by Hoff and Soong [17] to check the accuracy of the earlier approximations. As DMV shell theory has limited reliability for configurations in which the hoop deformation wavelengths are relatively large, the computations were repeated by using more refined equations based on Sanders' nonlinear kinematics [18]. Although the agreement between the two numerical approaches was quite encouraging, Simmonds & Danielson [19] pointed out that Hoff & Soong [17] failed to include all the pre-stress terms in Sanders' equations, and as a result those incorrect equations were hardly any more accurate than the shallow-shell theory used previously. Unfortunately, many authors in the open literature who claim to use Sanders' shell equations seem to have fallen in the same trap (e.g., [20]). The problems studied in [14, 15, 17] were also the subject of some investigations by Durban & Libai [21, 22], who used a special type of solution with separable variables in order to obtain (tight) lower and upper bounds for the buckling loads. A more detailed review of the work by Hoff and his associates is given in references [8] (pp. 81–85) and [9] (pp. 875–897).

In practice, the axial compression of thin elastic cylinders is usually carried out by stiffening the edges of the shells (e.g., using rigid rings or circular plates). As early as 1932-1934, Flügge (see [23], pp.252–258) hypothesised that rigorous buckling analyses for these structures would have to account for the rotations near the shell edge in the basic state – see Figure 1. The linear membrane stress state is valid only for infinitely long cylinders or for a Poisson's ratio equal to zero. However, Flugge's observation had to wait for almost three decades, before the development of numerical solutions was sufficiently advanced to enable the numerical integration of the relevant buckling boundary-value problem with *variable* coefficients. Stein [24, 25] provided the first acurate such finite-differences solution for the DMV cylinder buckling equations in displacements, with a basic state that incorporated the effect of axi-symmetric bending. His work was limited to simply-supported cylinders having the circular edges free to move in the tangential direction, but the effect of an external pressure was also considered (in addition to the axial compression). Fischer [26, 27] employed different in-plane boundary conditions to solve the same problem, but obtained different critical loads. To clarify the discrepancies, Almroth [28] re-visited these earlier studies by performing a more comprehensive numerical investigation vis-à-vis the role played by the edge restraints. To this end, eight different sets of boundary conditions were considered, with the shell edges being supported in the radial direction in each one of them. These correspond to the end constraints used by Hoff *et al.*, and in this work we shall also adopt a subset of them (as explained in §2.1). Rather unexpectedly, in all of the studies just mentioned it was discovered that the behaviour of the cylinders was rather similar to the corresponding scenarios involving a uniform membrane pre-buckling state of stress. As the axisymmetric bending considered by Stein, Fischer, and Almroth depends nonlinearly on the loading parameter, this is not an immediately obvious feature that could be anticipated *a priori*. It is one of the main goals of the present study to throw further light upon the link between the two sets of problems by using a mixture of perturbation techniques and direct numerical simulations of the original DMV bifurcation equations.

Extensions of the "rigorous" buckling analyses mentioned in the previous paragraph to circular

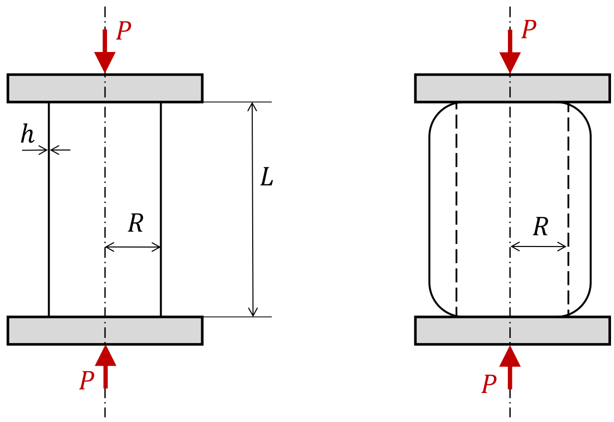

Figure 1: Axially symmetric deformation for a uniformly compressed circular cylindrical shell. The undeformed configuration appears on the left, while the ensuing "barelling" state is sketched on the right (greatly exaggerated/simplified for illustration purposes only).

cylindrical shells with different types of anisotropy and composites have been considered by many authors. This topic falls beyond the scope of our study, and we mention here only a couple of such examples. Stavsky & Friedland [29] re-worked Hoff's analysis [11] for the axisymmetric buckling of a heterogeneous, orthotropic circular cylindrical shell, while Gavrilenko & Stepanenko [30] included the possibility of asymmetric instability deformations for two particular types of edge supports in the case of an orthotropic circular cylinder. The latter authors also carried out extensive investigations into the buckling of ring-reinforced cylindrical shells, a scenario in which the nonlinear axisymmetric bending solution still remains relevant.

A word about the choice of equations used in the present study is in order. The DMV buckling equations employed in the rest of the paper have certain limitations that are well understood and widely documented. For example, Yamaki [7] has provided an extensive comparative *numerical* study of several sets of bifurcation equations for thin circular cylinders, by using both the classical membrane basic state and the more rigorous nonlinear bending solution mentioned above. Evidence included in the aforementioned reference confirms that the predictions of the DMV buckling equations are in very good agreement with those of the more accurate Flügge's equations [23], except for very short or very long cylinders. In light of these remarks, we shall confine our attention to cylinders of medium length (in a sense made clear in [7]). Further evidence that the DMV equations are a legitimate choice (with the caveats outlined above) is also provided by the results reported by Simmonds & Danielson in [19]. Their analysis was based on a set of equations derived previously in [31], whose accuracy is comparable to the Koiter-Sanders-Budiansky shell theory (e.g., see [18, 32, 33, 34]). Almost identical buckling equations for shells of revolution – and subject to less restrictive assumptions, were derived by Barta around 1966 and were later reported in [35]. In the absence of pre-stress, and depending on the geometry of the shell midsurface, these reduce to Morley's equations for cylinders [36] or to Vlasov's equations for a complete sphere [37] (pp. 612-619, eqns. (15.10)). A detailed analysis of Barta's equations for the buckling of an axially compressed cylindrical shell was given by Ivan [38, 39], who found that the predictions of the critical loads were very close to those obtained via Flügge's much more complicated equations. Unfortunately, neither [31] nor [35] can be easily adapted to cope with variable pre-stresses as they were originally derived only for uniform initial stress states.

With this background in mind, one of the main goals of the present paper is to revisit the classical works of Hoff et al. [11, 14, 15] by relaxing the original assumptions regarding the uniform nature of the pre-buckling deformations, and by exploring the use of singular perturbation methods. In contrast to those earlier investigations, here we adopt a more rigorous nonlinear pre-buckling solution that allows us to enforce more consistently the boundary constraints on both the pre-buckling equilibrium solution as well as the buckling eigenmodes. This is not the case for the uniform membrane solution, for which there are no boundary conditions. In fact, in the classical theory of buckling for thin cylindrical shells the edges are free of contraint and are thus free to move both axially and radially. The sketch included in Figure 2 displays the differences between the two aforementioned pre-buckling equilibrium solutions for a shell with simply supported edges. As seen there, the equilibrium radial nonlinear deformation developed as the result of the applied compression has quite a pronounced variation near the edge, but attenuates quite rapidly to match the membrane pre-buckling deformation that predominates in the rest of the shell. We make use of this asymptotic behaviour in order to simplify the corresponding eigenvalue problems. In the case of two particular types of boundary constraints we actually derive closed-form expressions for the critical load of the cylinder which extend the earlier results obtained by Hoff and his associates.

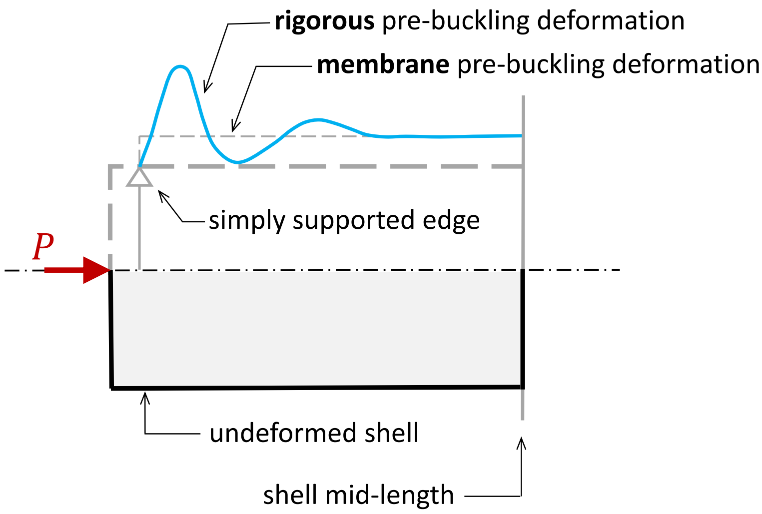

Figure 2: Pre-buckling deformation for an axially compressed thin elastic cylinder: membrane vs. nonlinear bending basic state (adapted from [40]).

The paper is laid out as follows. We start off in §2 with an overview of the main bifurcation problem, which consists of the classical DMV system coupled with a variable axisymmetric basic state. The boundary conditions adopted in this study are discussed separately in §2.1 for easy reference. These end constraints give rise to four distinct bifurcation problems, which for convenience are identified as P1–P4. A brief discussion of the particular basic state adopted in our investigations is outlined in §3. While the closed form-solution recorded there has been known for some time (being originally derived by Föppl [41]), the asymptotic simplification we propose does not seem to have been used before. In §4 we start our discussion of the problems P1 and P2, which share a number of similarities. By taking advantage of the localised eigendeformations in these two problems, we indicate an asymptotic simplification that allows us to identify the critical buckling load via a simple numerical strategy based on the simplified equations. The other two problems (P3 and P4) require a change of tack. Before the relevant changes are implemented, some key numerical results pertaining to these scenarios are discussed in §5.1. We then carry out a simple perturbation analysis for problem P4, and show that the two-term formula obtained there performs remarkably well when compared to the direct numerical simulations of the full DMV bifurcation equations. It turns out that this perturbation strategy must be modified in the case of P3, as this last case involves an unusual type of singular perturbation problem. Finally, the paper concludes with a number of remarks and ideas for possible extensions of this work.

2 Outline of the bifurcation equations

We consider a thin cylindrical shell of length L, radius R and uniform thickness h $(0 < h/R \ll 1)$ subjected to compressive axial forces $P > 0$ – see Figure 1. Its geometry is specified by the axial coordinate 'x' and a circumferential arc-length coordinate $s > 0$ that runs along the median curve of the transverse cross-section.

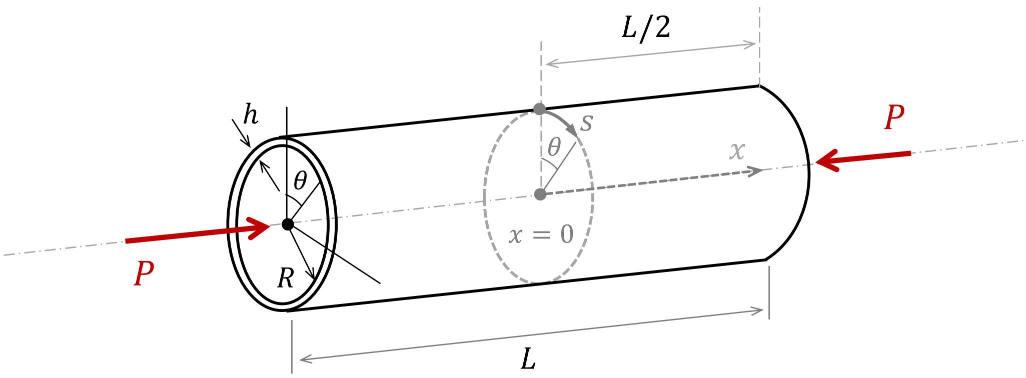

Figure 3: The geometry of a thin cylindrical shell of length L and radius R compressed by uniform axial forces $P > 0$.

The starting point for setting up the relevant bifurcation problem is the well-known Donnell-Mushtari-Vlasov (DMV) shallow-shell nonlinear equations (e.g., see [42, 43]) formulated in terms of the transverse displacement $w \equiv w(x, s)$ and a stress function $f \equiv f(x, s)$,

$$D\nabla^4 w + R^{-1} f_{,xx} - [f, w] - p = 0\,, \tag{2.1a}$$

$$(Eh)^{-1}\nabla^4 f - R^{-1} w_{,xx} + [w, w] = 0\,, \tag{2.1b}$$

where $D \equiv Eh^3/12(1-\nu^2)$ represents the bending rigidity of the cylinder, the subscripts preceded by a comma indicate (partial) differentiation with respect to the corresponding independent variables, and the classical differential operators that feature in (2.1) correspond to

$$\nabla^2(\dots) := (\dots)_{,xx} + (\dots)_{,ss}\,, \qquad \nabla^4(\dots) \equiv \nabla^2\nabla^2(\dots)\,,$$

$$[\varphi, \phi] := \varphi_{,xx}\,\phi_{,ss} - 2\varphi_{,xs}\,\phi_{,xs} + \varphi_{,ss}\,\phi_{,xx}\,,$$

for any two arbitrary smooth functions $\varphi \equiv \varphi(x, s)$ and $\phi \equiv \phi(x, s)$. The parameter p in (2.1a) reflects the effect of a uniform transverse pressure (acting either internally or externally).

For axisymmetric deformation of the compressed cylindrical shell the transverse displacement is a function of the axial coordinate x only. Letting $w_0 \equiv w_0(x)$ be such a lateral displacement, with the corresponding stress function f_0, it is known (e.g., see [6, 8]) that the nonlinear system (2.1) can be reduced to just one ordinary differential equation,

$$Dw_{0,\,xxxx} + Nw_{0,\,xx} + \left(EhR^{-2}\right) w_0 - \left(\nu NR^{-1} + p\right) = 0\,, \tag{2.2}$$

with $N \equiv P/(2\pi R)$ being the force per unit of circumference, uniformly distributed along the edge of the cylinder. For the sake of completeness, we mention that f_0 depends on both x and s, and satisfies

$$f_{0,\,xx} = \left(EhR^{-1}\right) w_0 - \nu N\,, \qquad f_{0,\,ss} = -N\,.$$

Bifurcations from the radially symmetric deformation mentioned above are readily obtained by employing the method of adjacent equilibrium. This involves writing $w \to w_0(x) + w_1(x, s)$ and $f \to f_0(x) + f_1(x, s)$, that are then substituted in the nonlinear equations (2.1), followed by the usual linearisation of the resulting equations. The outcome is the following system,

$$D\nabla^4 w_1 + R^{-1} f_{1,\,xx} + Nw_{1,\,xx} - \underline{w_{0,\,xx} f_{1,\,ss}} + \left[\nu N - \left(EhR^{-1}\right) w_0\right] w_{1,\,ss} = 0\,, \tag{2.3a}$$

$$(Eh)^{-1}\nabla^4 f_1 - R^{-1} w_{1,\,xx} + \underline{w_{0,\,xx} w_{1,\,ss}} = 0\,, \tag{2.3b}$$

in which some of the coefficients depend on $w_0(x)$ and its second-order derivative. The classical theory of buckling for axially compressed cylindrical shells is obtained by adopting a membrane basic state in which the underlined terms above are dropped; this corresponds to the *constant* solution

$$w_0 = R(Eh)^{-1}(\nu N + pR)\,.$$

Note that in the absence of the pressure term, the last term in equation (2.3a) will also drop out in the classical theory. As we are primarily interested in the effect of the axial compression we shall set $p \equiv 0$ in the pre-buckling equation (2.2).

It is useful to re-scale (2.3) by introducing the non-dimensional quantities

$$\overline{x} := \frac{2x}{L}\,, \quad c := \left[3(1-\nu^2)\right]^{1/2}, \quad \overline{w}_0 := c\left(\frac{w_0}{h}\right), \quad \overline{w}_1 := c\left(\frac{w_1}{h}\right), \quad \overline{f}_1 := \frac{f_1}{D}\,, \tag{2.4a}$$

$$\alpha := \frac{\sqrt{3}}{4}(1-\nu^2)^{1/2}\frac{L^2}{Rh}\,, \qquad N_{\text{cr}} := \frac{Eh^2}{cR}\,, \qquad \Lambda := \frac{N}{N_{\text{cr}}}\,, \tag{2.4b}$$

and we remark that the circumferential coordinate 's' can be replaced with the polar angle 'θ' since $s = R\theta$ (see Figure 3). To avoid complicating the notation unnecessarily, in what follows we shall drop the bar on x; thus, $|x| \le 1$ and the length of the re-scaled cylinder will then be equal to 2. We note in passing that our α above is proportional to the so-called *Batdorf* (or *curvature*) *parameter* – see [44], $Z \equiv (1-\nu^2)^{1/2}L^2/(Rh)$, which is used extensively in the literature on circular cylindrical shells (e.g., see [6, 7, 45, 46]). The quantity $N_{\text{cr}} \equiv \sigma_{cr}h$ defined in (2.4b) is the critical membrane force associated with formula (1.1).

The bifurcation system (2.3) is easily re-scaled in terms of the dimensionless quantities defined in (2.4), but in the interest of brevity we leave out those routine manipulations. Instead we simply point out that solutions with separable variables of the corresponding non-dimensional equations will be sought in the form

$$\overline{w}_1(x,\theta) = W(x)\sin(m\theta)\,, \qquad \overline{f}_1(x,\theta) = F(x)\sin(m\theta)\,, \tag{2.5}$$

with the arbitrary integer $m > 0$ being determined subject to the requirement that it should render the global minimum of the curve $\Lambda = \Lambda(m)$, while all the other parameters are kept fixed. Performing the requisite substitutions, it turns out that the unknown amplitudes in (2.5) satisfy the linear system with variable coefficients,

$$W'''' - 2(\beta^2 - 2\alpha\Lambda)W'' + \beta^2\big\{\beta^2 - 4\alpha[\nu\Lambda - \overline{w}_0(x)]\big\}W + \alpha F'' + \beta^2\overline{w}_0''(x)F = 0\,, \tag{2.6a}$$

$$F'''' - 2\beta^2 F'' + \beta^4 F - 4\alpha W'' - 4\beta^2\overline{w}_0''(x)W = 0\,, \tag{2.6b}$$

where the dash is used for indicating differentiation with respect to $x \in [-1, +1]$ and

$$\beta := mL/(2R) > 0 \tag{2.7}$$

is the *scaled* mode number (which no longer is necessarily an integer).

The critical eigenvalue, Λ_c, and the critical mode number, m_c, are defined by the requirements

$$\Lambda_c = \min_{m>0} \Lambda(m;\alpha)\,, \qquad \text{with} \quad \Lambda_c = \Lambda(m_c;\alpha)\,. \tag{2.8}$$

In light of (2.7) it should be clear that one can define a critical β, by simply writing $\beta_c := m_c(L/(2R))$. Typically, elastic stability problems amenable to the separation of variables considered above will display a parabola-like dependence for Λ as a function of m (or β), and the minimization problem (2.8) is completely unambiguous. However, this is not case for some of the scenarios considered later in this paper. More specificaly, for certain boundary conditions it is found that the foregoing dependence is in fact monotonic increasing, which suggests that the critical value Λ_c will correspond to the smallest admissible value of m (from a physical point of view). The choice $m = 0$ in (2.5) leads back to an axisymmetric deformation, so the bifurcated solution is qualitatively the same as the basic state (as will be also confirmed shortly by theoretical arguments in §5). Clearly, the eigenvalues of (2.6) associated with $m = 0$ do not represent true bifurcation points and can thus be omitted. The value $m = 1$ corresponds to a simple rigid-body translation of the transverse cross sections; whether or not this value must be considered is a contentious issue. Some authors (e.g., Yamaki [7]) have regarded this choice as valid, arguing that it corresponds to a beam-like deformation of the cylinder. However, for this statement to be true the transverse displacement W in (2.5) must resemble (more or less) one-half of a sine wave. This is not true in some of the cases he considered, particularly for moderate values of (L/R) and $\alpha \gg 1$ – the situation we are mostly interested in. A number of other investigators have rejected $m = 1$, e.g. Stein [25], Hoff and his associates [14, 15], as well as Almroth [28]. In their studies the mode number is assumed to satisfy $m \geq 2$, and we note that the case $m = 2$ corresponds to a change of the original circular cross section into an elliptic one. We shall elaborate more on these issues later in §5.

Returning now to the two coupled equations (2.6), it is noted that they must be solved subject to appropriate boundary conditions as discussed next. Taking into account the symmetry of the cylinder geometry and the particular loading scenario, one would expect the displacements experienced by our configuration to be either symmetric (even) or anti-symmetric (odd) about the midpoint of the shell axis. This observation permits us to integrate (2.6) over the reduced range $0 \leq x \leq 1$. By choosing the axial coordinate to be zero at the point situated half-way between the ends of the cylinder, the symmetry conditions require the following additional requirements for the amplitudes W and F; for *even* eigendeformations:

$$W' = W''' = 0 \qquad \text{and} \qquad F' = F''' = 0 \qquad \text{at } x = 0\,, \tag{2.9}$$

while for the *odd* ones:

$$W = W'' = 0 \qquad \text{and} \qquad F = F'' = 0 \qquad \text{at } x = 0\,. \tag{2.10}$$

In this work we shall restrict attention to the case of simply supported ends in which the boundary conditions $w = w_{,xx} = 0$ at $x = \pm L$ translate to requiring

$$W = W'' = 0 \qquad \text{at } x = \pm 1\,. \tag{2.11}$$

Further conditions at the ends of the cylinders can take a number of forms; here we will explore four possibilities that we discuss next.

2.1 Boundary conditions at $x = 1$

Let N_x, N_s and N_{xs} be the in-plane membrane forces in the shell, and suppose that u and v represent the corresponding in-plane displacements in the axial and the circumferential directions, respectively. It is recalled that the classical result (1.1) is valid only if the edges of the cylinder satisfy $w = w_{,xx} = u_{,x} = v = 0$ for $x = \pm(L/2)$ and $Z > 2.85$ (e.g., [6, 44]).

Motivated by the earlier work of Hoff et al. ([14, 15]), in the remaining of this paper we will investigate cylinders for which the boundary conditions (2.11) are augmented with one of the following four choices for the in-plane quantities: (i) $u = v = 0$; (ii) $N_x = v = 0$; (iii) $u = N_{xs} = 0$ and (iv) $N_x = N_{xs} = 0$. These are standard work-conjugate boundary conditions that arise from the variational derivation of the DMV equations. The classical theory that predicts (1.1) is obtained for the second choice. For a detailed discussion of the mechanical interpretation of these constraints we refer to the textbook by Jones [40] (pp. 476-478), which includes also their counterparts for flexurally clamped edges.

Early studies on buckling of cylinders (see Timoshenko's excellent classical account [1]) relied exclusively on the assumption that the ends of the cylinder were free to deform as they were axially compressed. In this case the buckling deformation has an inextensible character and the displacements can be taken in the form of suitable doubly infinite series of trigonometric functions. Given the local nature of boundary conditions, for a long time it was believed that these conditions do not influence the buckling load for thin and long cylinders. Discrepancies between theoretical predictions and experiments have contributed to a re-examination of the possible sources for such a mismatch (e.g., [23]). Stein [24] and Fischer [26] seem to be the first to experiment with different types of end constraints in their computations of buckling loads for thin elastic cylinders.

Since the DMV equations (2.1) operate only with the transverse displacement (w) and the stress function (f), the usual constraints involving the in-plane displacments u and v must be converted in terms of those quantities. In particular, the conditions $v = 0$ and $u = 0$ at the right end of the cylindrical shell ($x = 1$) are equivalent to $F'' + \nu\beta^2 F = 0$ and $F''' - (2+\nu)\beta^2 F' - 4\alpha W' = 0$ respectively, where F is as defined in (2.5). This then yields four separate problems in which the requirements $W(1) = W''(1) = 0$, as stated in (2.11), are supplemented by

$$F'' + \nu\beta^2 F = 0, \qquad F''' - (2+\nu)\beta^2 F' - 4\alpha W' = 0\,; \tag{2.12a}$$

$$F'' = 0, \qquad F = 0\,; \tag{2.12b}$$

$$F' = 0, \qquad F''' - 4\alpha W' = 0\,; \tag{2.12c}$$

$$F = 0, \qquad F' = 0\,. \tag{2.12d}$$

We shall subsequently refer to the problems associated with (2.12a-d) as P1–P4, respectively. It turns out that the first pair of problems share some structural similarities, as do the other two cases. Therefore in it convenient to discuss these issues in pairs and this is done in sections §4 and §5 below.

3 The basic state

The governing equation for the pre-buckling deformation (2.2) can be expressed in terms of the non-dimensional quantities (2.4), and takes the form

$$\mathcal{L}_\Lambda[\overline{w}_0] = 4\nu\alpha^2\Lambda\,, \qquad \text{with} \qquad \mathcal{L}_\Lambda \equiv \frac{d^4}{dx^4} + 4\alpha\Lambda\frac{d^2}{dx^2} + 4\alpha^2\,. \tag{3.1}$$

We re-iterate that $\Lambda > 0$ represents a (non-dimensional) parameter that controls the intensity of the applied compression (relative to $N_{\rm cr}$), and the dependence of $\overline{w}_0$ on Λ is *nonlinear*. Extensive numerical work carried out by Yamaki and his associates (see [7], pp.104–140) suggests that the critical value of the load ratio Λ is less than 1 when $\alpha \gtrsim 100$, and we shall confine our attention to this regime henceforth.

Equation (3.1) is linear and has constant coefficients, so an analytical form of its solution is readily derived. When $0 < \Lambda < 1$ we find that

$$\overline{w}_0(x) = \nu\Lambda\big[1 + A_1\sin(a_1x)\sinh(a_2x) + A_2\cos(a_1x)\cosh(a_2x)\big]\,, \tag{3.2}$$

where $a_j^2 := \alpha\omega_j^2$ $(j = 1, 2)$ with $\omega_1 := \sqrt{1+\Lambda}$ and $\omega_2 := \sqrt{1-\Lambda}$. The constants $A_j \in \mathbb{R}$ $(j = 1, 2)$ are determined by the simply-supported requirements $\overline{w}_0(1) = \overline{w}_0''(1) = 0$, while elementary algebraic manipulations of these constraints indicate that

$$A_1 \equiv A_1^s := -\frac{2(\omega_1\omega_2)\sin a_1\sinh a_2 + (\omega_1^2 - \omega_2^2)\cos a_1\cosh a_2}{2\omega_1\omega_2(\cosh^2 a_2 - \sin^2 a_1)}\,, \tag{3.3a}$$

$$A_2 \equiv A_2^s := -\frac{2(\omega_1\omega_2)\cos a_1\cosh a_2 - (\omega_1^2 - \omega_2^2)\sin a_1\sinh a_2}{2\omega_1\omega_2(\cosh^2 a_2 - \sin^2 a_1)}\,. \tag{3.3b}$$

Unfortunately, the closed-form expression (3.2) is of limited practical use since the formulae (3.3) are rather unwieldy. As we will ultimately be mainly interested in the *asymptotic limits* of the bifurcation system (2.6), we can take advantage of the presence of the parameter α in (3.1). If $\alpha \gg 1$, simple scaling arguments suggest that $\overline{w}_0$ consists of an $\mathcal{O}(1)$ constant part together with a pair of $\mathcal{O}(\alpha^{-1/2})$ boundary layers situated near $x = -1$ and $x = 1$, respectively. Symmetry considerations assure us that we need only be concerned with one of these layers. For example, near $x = 1$ one can introduce the stretched coordinate X defined by

$$x = 1 - \alpha^{-1/2}X\,, \qquad X = \mathcal{O}(1)\,, \tag{3.4}$$

so $\overline{w}_0 \equiv \overline{w}_0(X)$ turns out to satisfy the boundary-layer equation

$$\frac{d^4\overline{w}_0}{dX^4} + 4\Lambda\frac{d^2\overline{w}_0}{dX^2} + 4\overline{w}_0 = 4\nu\Lambda\,. \tag{3.5}$$

The solution of (3.5) with the usual exponential decay away from the edge $X = 0$ yields the expression

$$\overline{w}_0 \equiv \overline{w}_0(X) = \nu\Lambda \left[1 - \frac{1}{\omega_1\omega_2} \cos(\omega_1 X + \Theta_s) \exp(-\omega_2 X) \right] , \tag{3.6}$$

in which the phase angle Θ_s is defined by the trigonometric relation

$$\sin\Theta_s = \Lambda\,, \qquad 0 < \Theta_s < \pi/2\,.$$

Formula (3.6) for the basic state in a simply supported cylinder represents an approximate result because the symmetry conditions at $x = 0$ are not satisfied exactly (but the errors are exponentially small when $\alpha \gg 1$). To get an idea about the accuracy of (3.6), we illustrate in Figure 4 several comparisons with the exact formula (3.2). It is clear that the agreement is excellent even for quite modest values of α. We also remark in passing that as Λ increases from 0 to 1 the edge deformation spreads over an increasing distance along the half-length of the cylinder. For $\Lambda = 1$, we have $\omega_2 = 0$ and $\overline{w}_0$ becomes a purely harmonic deformation.

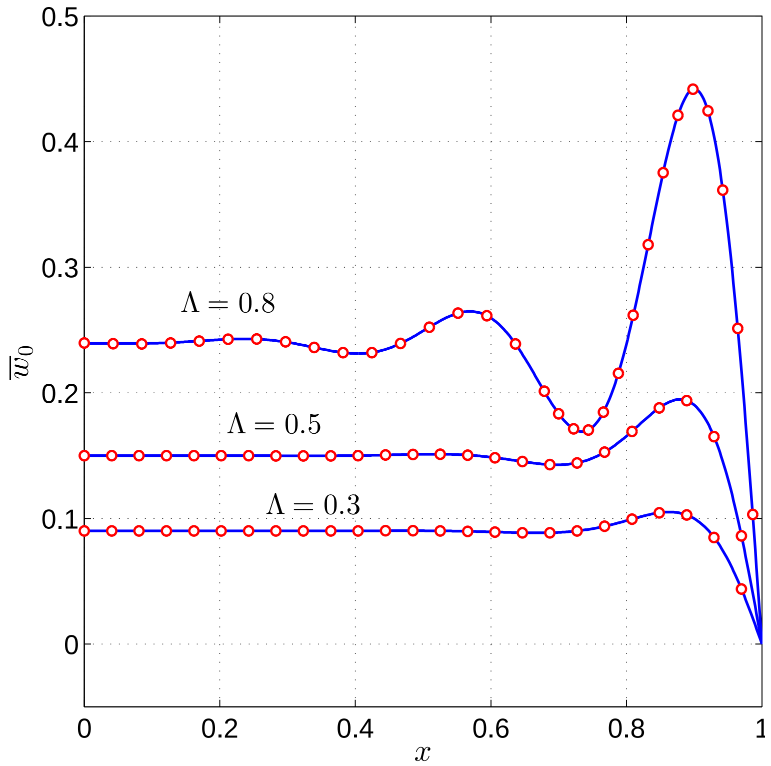

Figure 4: Comparison between the asymptotic approximation of the basic state (3.6) and the exact formula (3.2) with A_j $(j = 1, 2)$ given by (3.3). The latter is shown as the continuous (blue) curve, while the former results correspond to the (red) circular markers. Here, $\alpha = 2 \times 10^2$, the values of Λ are recorded next to the curves, and $\nu = 0.3$.

Equation (3.6) can be used to find simple approximations for the values of $\overline{w}_0'(x)$ and $\overline{w}_0'''(x)$ at

$x = 1$, in the limit $\alpha \gg 1$. In particular we note that

$$\overline{w}_0'\big|_{x=1} \simeq -\frac{\nu\Lambda\alpha^{1/2}}{\sqrt{1-\Lambda}}\,, \qquad \overline{w}_0'''\big|_{x=1} \simeq \frac{2\nu\Lambda\alpha^{3/2}}{\sqrt{1-\Lambda}}\,, \tag{3.7}$$

results that will be required below in our later analysis.

4 The problems P1 and P2

In this section we will examine a thin elastic cylinder with simply supported ends, corresponding to the problems designated as P1 and P2 in §2.1. In the works of Hoff and his associates (e.g., see [14, 15]) these problems were labeled SS4 and SS3, respectively. We recall that those studies were based on a constant basic state, which resulted in no critical eigenvalues with $0 < \Lambda_c < 1$ (i.e., no reduction of the buckling load from the classical theory was possible). Here, by contrast, the presence of variable pre-buckling deformations alters this conclusion as discovered by others in earlier numerical investigations (e.g., [28]). Our discussion below establishes the asymptotic structure of P1 and P2, and highlights some features of these problems that have hitherto remained obscure.

4.1 Numerical results

In order to appreciate the properties of the solutions we first obtained some numerical results derived directly from the bifurcation equations. For a given value of α, the eighth-order system (2.6) was solved with the basic state $\overline{w}_0(x)$ given by the expression (3.2), symmetry conditions applied at $x = 0$ and the simply supported requirements $W = W'' = 0$ imposed at $x = 1$. The last pair of boundary conditions at $x = 1$, as given by (2.12a) for P1 or (2.12b) for P2, then defines completely an eigenproblem for the loading parameter Λ (as a function of β and α).

Some sample results are illustrated in Figures 5 and 6. In Figure 5 we see the form of $\Lambda \equiv \Lambda(\beta;\alpha)$ for several values of α; it is evident that the overall shape of the solution curve is remarkably similar as α increases, and that the critical value of Λ appears to be almost independent of this quantity. The presence of the two local minima in the curves included in the right window of Figure 5 is not a numerical artefact, and appears to be a robust feature of the bucking equations themselves. A similar phenomenon was noted by Singer et al. [9] (Fig. 11.33, page 894) for a related situation in which the ends of the cylinder were taken to be clamped.

An inspection of the corresponding modal structure, as recorded in Figure 6, shows that as α increases so the eigensolutions are compressed against the end $x = 1$. This is confirmed by a simple asymptotic analysis that we describe next.

4.2 Asymptotic analysis for $\alpha \gg 1$

We have already seen in §3 that for large α the basic state takes on a two-layer structure. Across the bulk of the region, where $x = \mathcal{O}(1) < 1$, $\overline{w}_0(x)$ is virtually constant and it is only in the edge region where $1 - x = \mathcal{O}(\alpha^{-1/2})$ that there is any significant variation. In terms of the boundary layer co-ordinate X, as defined in (3.4), the basic state is given by (3.6).

The numerical evidence summarised above indicates that when $\alpha \gg 1$ the critical eigenmodes associated with problems P1 and P2 are confined to the region where $X = \mathcal{O}(1)$ and that the critical value of β is proportional to $\alpha^{1/2}$. Guided by this observation we put $\beta = \alpha^{1/2}B$ for some $B = \mathcal{O}(1)$, and expand the unknown functions and eigenvalue so that

$$W = W_0(X) + \dots\,, \qquad F = F_0(X) + \dots\,, \qquad \Lambda = \Lambda_0 + \dots\,, \tag{4.1}$$

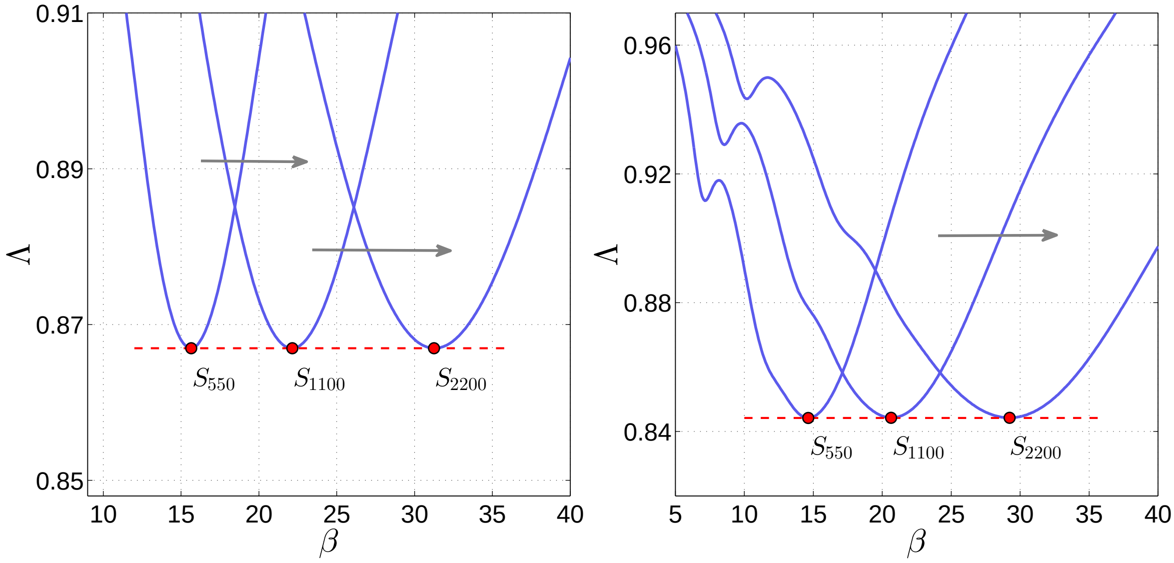

Figure 5: Samples of curves $\Lambda = \Lambda(\beta; \alpha)$ for P1 (left) and P2 (right) corresponding to $\alpha = 550,\ 1100,\ 2200$; in both cases the arrow indicates the direction in which α increases and the red markers S_α identify the location of the *critical point* (β_c, Λ_c).

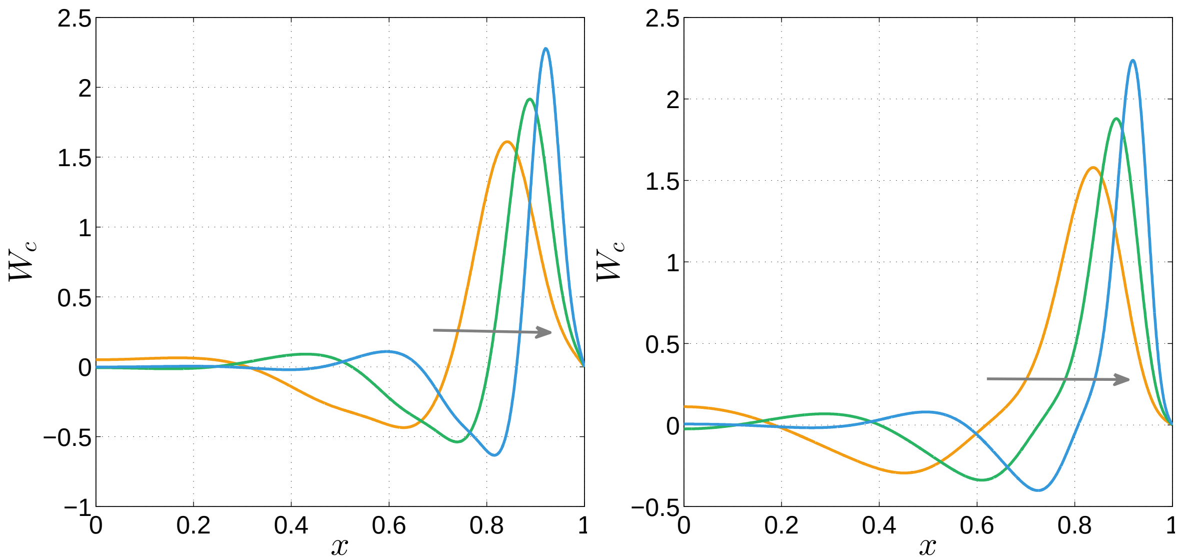

Figure 6: Sequences of *critical eigenmodes* corresponding to the red markers S_α ($\alpha = 550,\ 1100,\ 2200$) in Fig. 5: P1 (left) and P2 (right). The arrows show the direction in which parameter α increases, and all solutions are normalised so that $\int_0^1 W_c^2(x)\,dx = 0.3$.

where W_0 and F_0 are leading order functions to be found. Substituting these expressions into (2.6) and retaining the dominant terms yields the system

$$W_0'''' - 2(B^2 - 2\Lambda_0)W_0'' + B^2 \left[B^2 - \frac{4\nu\Lambda}{\sqrt{1-\Lambda^2}} \cos(\omega_1 X + \Theta_s) \exp(-\omega_2 X) \right] W_0 + F_0'' + B^2 \overline{w}_{0XX} F_0 = 0\,, \tag{4.2a}$$

$$F_0'''' - 2B^2 F_0'' + B^4 F_0 - 4W_0'' - 4B^2 \overline{w}_{0XX} W_0 = 0\,, \tag{4.2b}$$

where

$$\overline{w}_{0XX} = -\frac{2\nu\Lambda_0}{\sqrt{1-\Lambda_0^2}} \sin(\omega_1 X) \exp(-\omega_2 X)$$

is simply the second derivative of the expression (3.6) with $\Lambda = \Lambda_0$. This system needs to be solved subject to $W_0 = W_0'' = 0$ at $X = 0$ together with either

$$F_0'' + \nu B^2 F = 0 \qquad \text{and} \quad F_0''' - (2+\nu)B^2 F_0' - 4W_0' = 0 \quad \text{at } X = 0 \qquad \text{(Problem P1)}$$

or

$$F_0'' = 0 \qquad \text{and} \quad F_0 = 0 \quad \text{at } X = 0\,, \qquad \text{(Problem P2)}$$

where the dash in these equations stands for differentiation with respect to the variable X introduced in (3.4). We also demand that the eigensolutions be confined to the thin boundary layer so that both W_0 and F_0 decay exponentially as $X \to \infty$.

It is noticed that the asymptotic problem (4.2) essentially replicates the original system and there is minimal simplification of the full forms. The problem can only be solved numerically and this yields a curve that gives Λ_0 as a function of B. This curve is virtually identical in shape to any of the eigen-curves already seen in Figure 5; for instance, in the case of P1 the minimum is found at at $(\Lambda_0, B) = (0.866944, 0.667)$, while for P2 this point corresponds to $(\Lambda_0, B) = (0.844264, 0.623)$. To better appreciate the high accuracy of the simplified problems we illustrate in Figure 7 a sample of representative comparisons with the data shown earlier in Figure 5.

5 The problems P3 and P4

In this section we explore the remaining two cases, that is the problems designated as P3 and P4 in §2.1. These correspond, respectively, to the cases SS2 and SS1 studied by Hoff and Rehfield [14] and Hoff [15], whose buckling models were limited to a standard linear membrane basic state.

5.1 Numerical results

Our immediate goal is to determine the dependence of Λ_c on $\alpha \gg 1$, for moderately long cylinders. To this end, we note that if we fix (R/h) then $(L/R) \propto \alpha^{1/2}$ – see (2.4b). Conversely, if (L/R) is fixed (and hence β in (2.7) depends only on $m \geq 2$), then $(R/h) \propto \alpha$. In this latter case, direct numerical integration of the bifurcation system (2.6) subject to the boundary constraints (2.12c) or (2.12d) shows that $\Lambda = \Lambda(m; \alpha)$ is a monotonic increasing function of m. As discussed at the end of §2, the critical mode numbers are either $m_c = 1$ or $m_c = 2$, and numerical evidence suggests that Λ_c is about the same for each of these values; by following Stein [25] and Almroth [28], we choose $m_c = 2$. Thus, if $\nu = 0.3$ then $\beta = (L/R) \simeq 0.1556\,\alpha^{1/2}$ for $(R/h) = 100$, while $(L/R) \simeq 0.0898\,\alpha^{1/2}$ for $(R/h) = 300$. By solving the corresponding bifurcation problems with these values of β and for a range of α-values we arrive at the curves presented in Figure 8, which capture the dependence of Λ_c on (L/R) (or, equivalently, on $\alpha^{1/2}$). It is clear both P3 and P4 display very similar behaviour, with

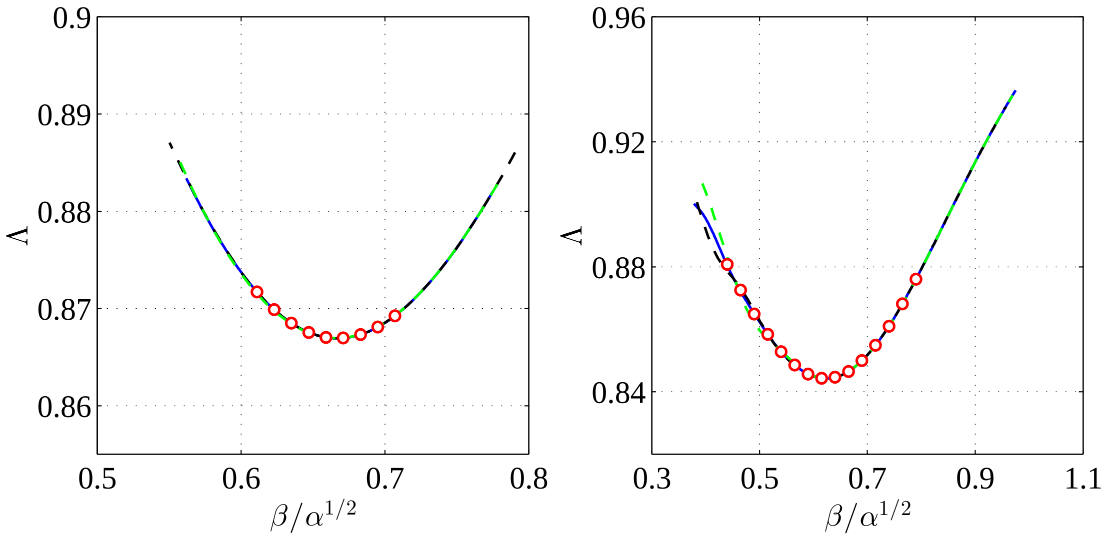

Figure 7: Comparisons between the predictions of the asymptotically simplified problems P1 and P2 – see the system (4.2), and the corresponding response curves recorded in Fig. 5. The data for P1 is included in the left window, while that for P2 appears in the right one. In both windows the red markers represent the asymptotic predictions, the response curves being "stacked" on top of each other and shown in blue ($\alpha = 2200$), black ($\alpha = 1100$), and green ($\alpha = 550$), respectively.

Λ_c in each case showing very little variation once $(L/R) \gtrsim 1.5$. For P3 the asymptotic values of the load ratio are $\Lambda_c \simeq 0.510$ ($R/h = 10^2$) and $\Lambda_c \simeq 0.503$ ($R/h = 3 \times 10^2$), while for P4 very similar values are obtained, with the corresponding differences being at most $\mathcal{O}(10^{-4})$.

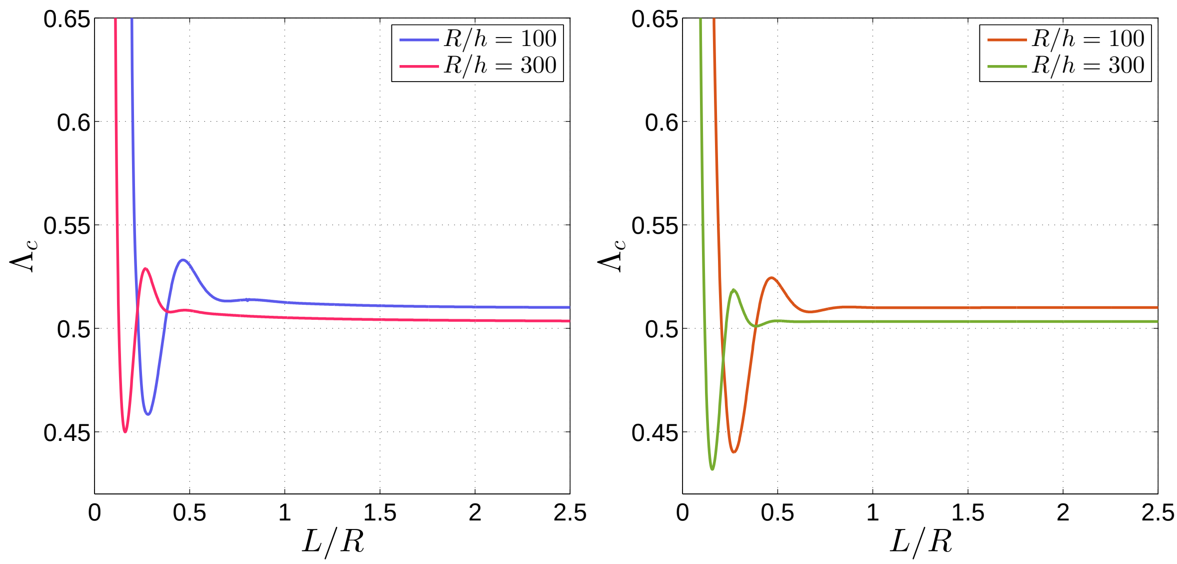

Figure 8: Typical dependence of the critical load ratio Λ_c on (L/R) for cylinders with $(R/h) \in \{100, 300\}$: P3 (left) and P4 (right). In all cases the buckling load becomes constant for sufficiently large aspect ratios.

Having seen the behaviour of Λ_c with $\alpha > 0$, we now consider the approximation of the above asymptotic values using some reduced bifurcation systems. Samples of curves $\Lambda = \Lambda(\beta; \alpha)$ for P3 and P4 are included in Figure 9, where each curve corresponds to a different value of α. The bifurcation equations (2.6) were solved by using the appropriate boundary conditions from §2.1, with α being kept fixed while $\beta > 0$ was varied in order to capture the desired dependence of Λ for $\alpha \gg 1$; for clarity purposes, in Figure 9 we have restricted the range of the scaled mode number to $0 \leq \beta \leq 1$. Note that in the case of P4, it appears that $\lim_{\beta \to 0^+} \Lambda_c = 1/2$ irrespective of the value of $\alpha \gg 1$. This contrasts to the behaviour of P3 for which the limit Λ_c as $\beta \to 0^+$ seems to be a function of α, with this limit approaching $1/2$ as $\alpha \to \infty$. We try to confirm some of these behaviours using formal asymptotic techniques which we outline now.

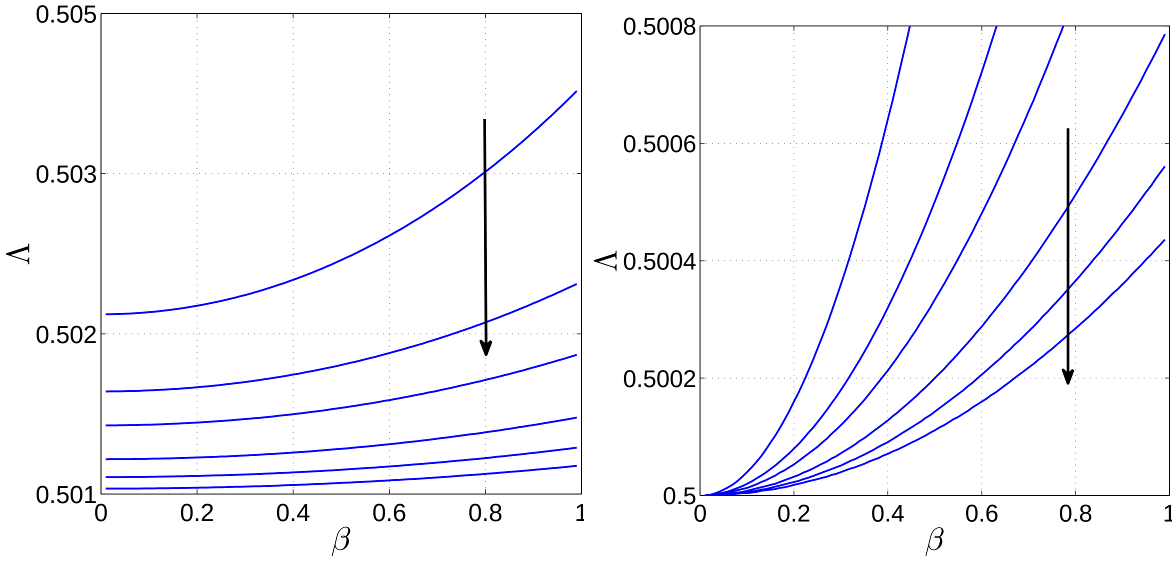

Figure 9: Samples of curves $\Lambda = \Lambda(\beta; \alpha)$ for P3 (left) and P4 (right) corresponding to $\alpha = j \times 10^2$, with $j \in \{1, 2, 3, 5, 7, 9\}$; in both cases the arrow indicates the direction in which α increases.

5.2 Asymptotic analysis for P4

We have already noted that for problem 4, is seems to be the case that Λ tends to exactly $1/2$ as $\beta \to 0^+$ whilst there appears to be a more complicated picture at hand in problem 3. For that reason we start our asymptotic analysis for the former case, and hope that an understanding of that underpinning structure will provide a guide to help us subsequently explain the more involved problem.

Before embarking on the analysis it is worth reminding the reader that when $\alpha \gg 1$ the basic state $\overline{w}_0$ takes on a two-zoned structure; across the majority of the cylinder $\overline{w}_0$ is a constant ($= \nu\Lambda$) and only exhibits significant variation in a thin layer of depth $\mathcal{O}(\alpha^{-1/2})$ adjacent to $x = 1$ where the co-ordinate $X = \mathcal{O}(1)$ (see (3.4)). If we were to follow standard asymptotic arguments we might attempt to develop solutions in the separate regimes where $x = \mathcal{O}(1)$ and where $X = \mathcal{O}(1)$, and then match the solutions together. In turns out, however, that in the current problem this is not the most advantageous strategy. Instead it proves possible to determine the eigensolution forms across the whole region $0 \leq x \leq 1$ and without the need to suppose that $\alpha \gg 1$ at the outset. This is the route we shall take and develop our solutions on the assumption that $\alpha = \mathcal{O}(1)$ and β is small. It is only when we need to determine expressions for the form of Λ when α is large that we will need to consider this limit explicitly.

It is helpful to write $\beta^2 = \alpha\varepsilon$, for $0 < \varepsilon \ll 1$, and seek a solution of the problem with

$$W = W_{04}(x) + \varepsilon W_{14}(x) + \dots, \qquad F = F_{04}(x) + \varepsilon F_{14}(x) + \dots, \qquad \Lambda = \Lambda_{04} + \varepsilon\Lambda_{14} + \dots; \quad (5.1)$$

the notation $\cdot_{04}$ denotes leading order quantities and $\cdot_{14}$ their corrections; we have included the second digit to indicate which problem is under consideration. If we substitute (5.1) into system (2.6), the leading-order terms give

$$W_{04}'''' + 4\alpha\Lambda_{04}W_{04}'' + \alpha F_{04}'' = 0, \qquad F_{04}'''' - 4\alpha W_{04}'' = 0\,, \quad (5.2)$$

where a dash denotes differentiation with respect to x. We commented earlier that the bifurcation system admitted families of solutions which are even-valued in x and others which are odd-valued. Our computations revealed that it is the former set which are the more important, in the sense that they arise at smaller loading, and we proceed under this assumption. One integration of (5.2b), and application of the symmetry gives $F_{04}''' - 4\alpha W_{04}' = 0$, while integrating again yields

$$F_{04}'' = 4\alpha(W_{04} - A)\,, \quad (5.3)$$

for some constant $A \in \mathbb{R}$ ($A \neq 0$). If we eliminate F_{04} from (5.2a), we are left with

$$W_{04}'''' + 4\alpha\Lambda_{04}W_{04}'' + 4\alpha^2 W_{04} = 4\alpha^2 A\,. \quad (5.4)$$

In turns out that this equation can be solved completely in terms of the basic state function $\overline{w}_0$ which we know satisfies (3.1) (and where we have put $\Lambda = \Lambda_{04}$ at leading order). We obtain the simple result

$$W_{04} = \frac{A}{\nu\Lambda_{04}}\overline{w}_0\,; \quad (5.5)$$

then it is possible to integrate (5.3) twice to deduce that

$$F_{04} = -\frac{A}{\nu\Lambda_{04}\alpha}\left(\overline{w}_0'' + 4\alpha\Lambda_{04}\overline{w}_0\right); \quad (5.6)$$

it is remarked that these leading order solutions automatically satisfy the desired symmetry conditions at $x = 0$ together with the relevant conditions that $W_{04} = W_{04}'' = F_{04} = 0$ at $x = 1$. All that needs to be ensured is that $F_{04}' = 0$ at $x = 1$. We can use the results (3.7) to deduce that

$$F_{04}'\big|_{x=1} = \frac{2A(2\Lambda_{04} - 1)}{\sqrt{1 - \Lambda_{04}}} \qquad \Longrightarrow \qquad \Lambda_{04} = \frac{1}{2}\,.$$

This confirms our earlier observation that $\Lambda \to 1/2$ as $\beta \to 0^+$, irrespective of the value of α.

We can now proceed to consider the correction terms. If we extract the $\mathcal{O}(\varepsilon)$ terms from the governing system (2.6) then we find that

$$W_{14}'''' + 4\alpha\Lambda_{04}W_{14}'' + \alpha F_{14}'' = 2\alpha(1 - 2\Lambda_{14})W_{04}'' - \alpha\overline{w}_0'' F_{04}, \quad (5.7a)$$

$$F_{14}'''' - 4\alpha W_{14}'' = 2\alpha F_{04}'' + 4\alpha\overline{w}_0'' W_{04}\,, \quad (5.7b)$$

which needs to be solved subject to the symmetry conditions at $x = 0$ and the requirements that $W_{14} = W_{14}'' = F_{14} = F_{14}' = 0$ at $x = 1$. Since equations (5.7) are just forced versions of (5.2), we need not solve these explicitly but rather just derive a suitable solvability condition. It is a standard procedure to derive the pair of functions that is adjoint to the system (5.2); these are $P(x) := W_{04}$ and $Q(x) := -\frac{1}{4}F_{04}$. If we multiply (5.7a) by $P(x)$, (5.7b) by $Q(x)$, add and then integrate, it follows that our problem for W_{14} and F_{14} only admits a solution if

$$(1-\Lambda_{14})\int_0^1 W_{04}'^2(x)\,\mathrm{d}x = -2\int_0^1 \overline{w}_0''(x)F_{04}(x)W_{04}(x)\,\mathrm{d}x\,.$$

If we use our solutions (5.5) and (5.6) for W_{04} and F_{04} respectively, then when $\alpha \gg 1$ we discover that $\Lambda_{14} = \frac{1}{4}(1+2\nu)$, whence

$$\Lambda = \frac{1}{2} + \frac{(1+2\nu)\beta^2}{4\alpha} + \dots\,. \tag{5.8}$$

A comparison of this formula against the numerical results given in Figure 9 is excellent; if this is superimposed on the curves shown the difference is imperceptible, at least for β over the range indicated – see Figure 10.

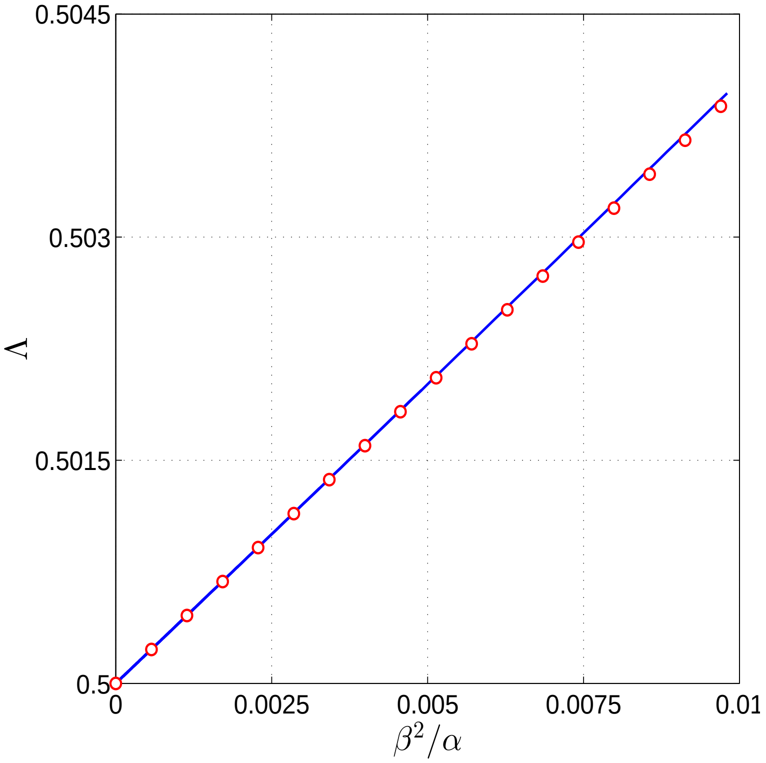

Figure 10: The predictions of our asymptotic result (5.8), shown as red markers, are superimposed on the sample of response curves recorded in the right window of Fig. 9. The latter set of results is seen as a single blue curve because we display the behaviour of Λ on (β^2/α) rather than β.

5.3 Analysis of solutions of P3 when $\alpha \gg 1$

We now turn to examine the remaining problem, which we have denoted P3. We have already pointed out that the numerical solutions of this case are in many ways similar to those for P4 in as much that the critical case seems to arise as $\beta \to 0^+$ and that this critical value is close to 1/2. On the other hand, while we have proved above that for P4 the response curves $\Lambda = \Lambda(\beta; \alpha)$ asymptote to 1/2 for all $\alpha \geq \mathcal{O}(1)$ as $\beta \to 0^+$, there is clear α-dependence on the limit of these curves in the case of P3. Moreover, while P3 may be well-defined at small non-zero values of β, the value $\beta = 0$ must be some kind of singular limit; we can conclude this because when $\beta = 0$ precisely, the bifurcation equation (2.6b) reduces to $F'''' - 4\alpha W'' = 0$; the requirements for even-valued solutions then give $F''' - 4\alpha W' = 0$ which is an exact duplication of the second of boundary conditions (2.12c) so that the boundary value problem is ill-defined.

Given the similarity between the numerical solutions of problems P3 and P4, it is plausible that the relevant solution structures ought to be closely related to each other. It is therefore tempting to propose a stucture similar to (5.1), viz.

$$W = W_{03}(x) + \varepsilon W_{13}(x) + \dots\,, \qquad F = F_{03}(x) + \varepsilon F_{13}(x) + \dots\,, \qquad \Lambda = \Lambda_{03} + \varepsilon \Lambda_{13} + \dots\,.$$

However, closer inspection of the correction equation (5.7b) reveals that the structure must need some revision. One integration of this equation shows that the boundary condition for P3 at $x = 1$ can only be met if $\int_0^1 \overline{w}_0''(x) W_{04}(x)\,\mathrm{d}x = 0$ and this is not the case; this means that the leading-order part of the W eigenfunction must be modified in some way. This can be achieved by introducing an asymptotically larger component into the form of the function F so that

$$W = W_{03}(x) + \varepsilon W_{13}(x) + \dots\,, \qquad F = \varepsilon^{-1} F_{-1}(x) + F_{03}(x) + \varepsilon F_{13}(x) + \dots\,, \tag{5.9a}$$

$$\Lambda = \Lambda_{03} + \varepsilon \Lambda_{13} + \dots\,. \tag{5.9b}$$

The leading-order equation involves only the $\mathcal{O}(\varepsilon^{-1})$-term in the expansion of F and this satisfies $F_{-1}'' = 0$; the solution which is even-valued about $x = 0$ is just

$$F_{-1}(x) \equiv K$$

for some constant $K \in \mathbb{R}$. The next-order equations follow very similar lines to those described in our account of P4; in particular we again find that F_{03} is related to W_{03} using

$$F_{03}'' = 4\alpha(W_{03} - A)\,, \tag{5.10}$$

for some constant $A \in \mathbb{R}$ ($A \neq 0$), while the fourth-order equation for W_{03} assumes the form

$$W_{03}'''' + 4\alpha\Lambda_{03} W_{03}'' + 4\alpha^2 W_{03} = 4\alpha^2 A - K\overline{w}_0''; \tag{5.11}$$

we point out that the additional term on the right-side side arises owing to the presence of F_{-1}. Now we have introduced two constants A and K and our aim is to relate them. Although equation (5.11) is more complicated than before, we are still able to solve it in terms of $\overline{w}_0$. Routine calculations show that

$$\widetilde{W}_{03}(x) = \gamma_0 \overline{w}_0(x) + x\left[\gamma_1 \overline{w}_0'(x) + \gamma_3 \overline{w}_0'''(x)\right] \tag{5.12}$$

is a particular solution of (5.11) where

$$\gamma_0 := \frac{A}{\nu \Lambda_{03}}\,, \qquad \gamma_1 := \frac{K\Lambda_{03}}{8(1 - \Lambda_{03}^2)}\,, \qquad \gamma_3 := \frac{K\alpha^{-1}}{16(1 - \Lambda_{03}^2)}\,.$$

In (5.12) the function $\overline{w}_0$ is the leading-order part of the basic state that results from expanding Λ according to (5.9b).

We need the solution of equation (5.11) that satisfies $W_0 = W_0'' = 0$ at $x = 1$, as stipulated by the original boundary conditions consistent with the simply supported edge constraints considered in this work. On noting that the functions $(\overline{w}_0 - \nu\Lambda_{03})$ and $\overline{w}_0''$ are themselves particular solutions of the homogeneous form of equation (5.11), we can add appropriate multiples of these to (5.12) to obtain the solution that has the requisite properties. This full solution turns out to be

$$W_{03}(x) = \widetilde{W}_{03}(x) + \gamma_2\overline{w}_0''(x) + \gamma(\overline{w}_0 - \nu\Lambda_{03})\,, \tag{5.13}$$

with

$$\gamma_2 := \frac{2\alpha^{-3/2} + \alpha^{-1}\sqrt{1-\Lambda_{03}}}{16(1-\Lambda_{03}^2)}\,, \qquad \gamma := -\frac{1}{8(1+\Lambda_{03})\sqrt{1-\Lambda_{03}}}\,.$$

In principle, we can use (5.10) to deduce the function F_{03} but this is unnecessary. It is sufficient to integrate just once to derive

$$F_{03}'(x) = \Gamma_3\overline{w}_0'''(x) + \Gamma_2\overline{w}_0''(x) + x\big[\Gamma_1\overline{w}_0'(x) + \Gamma_0\overline{w}_0(x) + \Gamma\big]\,,$$

with

$$\Gamma_3 := -\frac{A\alpha^{-1}}{\nu\Lambda_{03}} + \frac{K\alpha^{-1}}{8\sqrt{1-\Lambda_{03}^2}}\left(\Lambda_{03} - \alpha^{1/2}\sqrt{1-\Lambda_{03}}\right)\,,$$

$$\Gamma_1 := -\frac{4A}{\nu} + \frac{K}{4(1-\Lambda_{03}^2)}\left[\alpha^{1/2}(1+2\Lambda_{03})\sqrt{1-\Lambda_{03}} + (3-2\Lambda_{03}^2)\right]\,,$$

$$\Gamma_2 := \frac{K}{4(1-\Lambda_{03}^2)}\,, \qquad \Gamma_0 := 2\alpha\Lambda_{03}\Gamma_2\,, \qquad \Gamma := -\nu\Lambda_{03}\Gamma_0\,,$$

and demand that $F_{03}'(1) = 0$. This yields a first relation between the constants A and K in the form

$$2A(1-2\Lambda_{03}) = \frac{\nu\Lambda_{03}K(3-2\Lambda_{03})}{4\alpha^{1/2}(1-\Lambda_{03})}\,. \tag{5.14}$$

A second connection is established by looking at the next order equation

$$F_{13}'''' - 4\alpha W_{13}'' = 2\alpha F_{03}'' - \alpha^2 F_{-1} + 4\alpha\overline{w}_0''W_{03}\,,$$

which when integrated once becomes

$$F_{13}''' - 4\alpha W_{13}' = 2\alpha F_{03}' - K\alpha^2 x + 4\alpha\int_0^x \overline{w}_0''(x)W_{03}(x)\,\mathrm{d}x\,. \tag{5.15}$$

The constant of integration has been chosen to ensure that F_{13} and W_{13} are even-valued functions. Now the left-hand side of this balance must be zero when evaluated at $x = 1$, as required by the boundary conditions specified in P3. Since we have demanded that $F_{03}'(1) = 0$ then

$$\int_0^1 \overline{w}_0''(x)W_{03}(x)\,\mathrm{d}x = \frac{1}{4}K\alpha\,. \tag{5.16}$$

Since we have already calculated W_{03} in (5.13), the integral constraint (5.16) can be reduced to a rather tedious exercise in integration by parts. It can be shown that all integrals involved can be written in terms of $\int_0^1 \overline{w}_0''^2(x)\,\mathrm{d}x$ and $\int_0^1 \overline{w}_0'^2(x)\,\mathrm{d}x$, which can be evaluated asymptotically for $\alpha \gg 1$. In particular, in this limit

$$\int_0^1 \overline{w}_0'^2(x)\,\mathrm{d}x \simeq \frac{\nu^2\Lambda_{03}^2(3-2\Lambda_{03})\alpha^{1/2}}{4(1-\Lambda_{03})^{3/2}}\,, \tag{5.17a}$$

$$\int_0^1 \overline{w}_0''^2(x)\,\mathrm{d}x \simeq \frac{\nu^2\Lambda_{03}^2\alpha^{3/2}}{2(1-\Lambda_{03})^{3/2}}\,. \tag{5.17b}$$

Evaluating (5.16) with the help of (5.17) leads to

$$\frac{\nu\Lambda_{03}(3-2\Lambda_{03})A}{4(1-\Lambda_{03})^{3/2}} = -K\left[\frac{1}{4}\alpha^{1/2} - \frac{\nu^2\Lambda_{03}^2(2\Lambda_{03}-5)}{16(1-\Lambda_{03})^{5/2}}\right]. \tag{5.18}$$

The eigenrelation for Λ_{03} follows from (5.14) and (5.18); these are consistent only if

$$2(1-2\Lambda_{03})\left[4(1-\Lambda_{03})^{5/2} - \nu^2\Lambda_{03}^2(2\Lambda_{03}-5)\alpha^{-1/2}\right] + \underline{\nu^2\Lambda_{03}^2(3-2\Lambda_{03})^2\alpha^{-1/2}} = 0\,. \tag{5.19}$$

In the limit $\alpha \to \infty$, the underlined term in this equation diminishes and the solution given by (5.19) is just $\Lambda_{03} = 1/2$. For $\alpha \gg 1$ the aforementioned term is small, so we expect that $\Lambda_{03} \equiv \Lambda_{03}(\alpha)$ will be close to 1/2. A routine perturbation analysis of (5.19) reveals that the root Λ_{03} of that equation admits an expansion of the type

$$\Lambda_{03} = \frac{1}{2} + \frac{\nu^2\alpha^{-1/2}}{2\sqrt{2}} + \frac{3\nu^4\alpha^{-1}}{8} + \mathcal{O}(\alpha^{-3/2})\,. \tag{5.20}$$

In Table 1 we have included a comparison between the values Λ_{03} obtained by numerically solving (5.19) and the direct numerical simulations of the full problem P3. Clearly, the agreement between the two sets of data is excellent. It is interesting to note that the three-term formula (5.20) reproduces exactly the values predicted in the second column of the table.

Table 1: Comparison between the asymptotic results (5.19) and the full numerical simulations of problem P3; the latter set of Λ-values corresponds to $\beta \simeq 0$ in the left window of Figure 9. The relative errors (*R.E.*) between the two sets of values are recorded in the last column.

α	Λ_0 (asymptotics)	Λ (full numerics)	*R.E.* ($\times 10^{-4}$)
100	0.503212	0.503244	0.6359
200	0.502265	0.502281	0.3185
300	0.501847	0.501858	0.2192
500	0.501429	0.501435	0.1197
700	0.501207	0.501211	0.0798
900	0.501064	0.501067	0.0599

6 Conclusions

We have investigated the compressive buckling of a simply supported thin elastic circular cylindrical shell subjected to four different types of in-plane conditions. The emphasis here has been on taking into account the axisymmetric *nonlinear bending deformation* experienced by the cylinder prior to the onset of buckling. Although the numerical analysis of the scenarios considered in this study has been known for some time (e.g., [28, 6, 7]), to the best of our knowledge this is the first time that a rational asymptotic structure of these problems has been elucidated in any detail using perturbation methods. In particular, we have been able to show that in the case of simply supported ends some of the earlier findings of Hoff et al. for the case of a uniform membrane basic state (discussed in §1), carry over to the nonlinear situation considered in this work and are amenable to an analytic description. Our main results consist of simple asymptotic predictions for the critical load ratio Λ_c (defined in (2.4b) and (2.8)) as a function of the Batdorf parameter α and the scaled mode number β – for the precise definitions of these quantities see (2.4b) and (2.7). Details of a couple of such formulae are recorded in (5.8) and (5.20). For the remaining two sets of in-plane boundary conditions considered, we have shown that the asymptotic simplification of the full DMV buckling system is only marginally simpler than the original; furthermore, in those cases the additional corrections to the leading-order Λ_c derived in §4 will offer no benefit since these leading-order predictions have only excluded exponentially small terms.

To illustrate the similarity between our result for problem P4 and Hoff & Rehfield's finding for its counterpart (SS1), we express both predictions for the critical load ratio in terms of the ubiquitous shell parameter (h/R). To this end, we substitute into (5.8) the expressions of α and β from (2.4b) and (2.7), respectively, and set $m = 2$. This results in $\Lambda_c = \Lambda_{BC}$, where the latter quantity is defined below,

$$\Lambda_{BC} := \frac{1}{2} + \frac{1+2\nu}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{R}\right) + \ldots .$$

Hoff & Rehfield's result follows from their equations (78) and (79) in [14], and reads $\Lambda_c = \Lambda_{HR}$, where

$$\Lambda_{HR} := \frac{1}{2} + \frac{2}{\sqrt{3(1-\nu^2)}}\left(\frac{h}{R}\right) + \ldots . \tag{6.1}$$

Clearly, if $\nu \simeq 0.3$ then the predictions of these formulae are very similar; in this case the presence of a (nonlinear) variable basic state has a marginal impact on the final result. A similar comparison for P3 is not available since our asymptotic analysis in §5.3 captured only the leading-order behaviour of Λ_c. Nevertheless, as our $\Lambda_{03} \simeq 1/2$ and reference [14] identified (6.1) as still being valid for their SS2 (whose our counterpart is P3), the agreement between the two sets of results is still very close.

It is perhaps worth emphasising that, even though the pre-bucking state is available in closed form, our arguments have not relied on its availability. In fact, our entire theoretical developments were carried out with the help of the boundary-layer approximation (3.6). A close inspection of the original bifurcation equations suggests that the approach taken here (in problems P3 and P4) can be adapted to the case of a uniform pre-buckling state as well. There are several key advantages in taking this particular route: (a) the complicated rescaling used by Hoff et al. becomes redundant; (b) there is no need to rely on closed-form exponential solutions and awkward determinantal equations.

Generally speaking, the DMV shell theory employed throughout this paper forms a robust framework for investigating non-symmetric deformations of cylinders, provided that the mode number m in (2.5) is sufficiently large (typically, $m \geq 4$). In our present analysis, as well as in the works of Hoff and Almroth discussed in §1, this requirement is violated, and this represents a limitation of the shallow-shell theory adopted. On the other hand, the extensive numerical results of Yamaki [7] show that, qualitatively, the drastic reduction of the buckling load in cases P3 and P4 is not simply an artefact of the equations used, as it is also encountered in models based on Flügge's buckling equations. It is interesting to note that in more recent times Yamaki's results were corroborated by even more sophisticated shell buckling equations (e.g., [47]). In future, it may be worth revisiting our analysis of P3 and P4 on the basis of a more complete set of bifurcation equations that is not limited to 'shallow modes'. Of course, this is likely to require non-trivial modifications of our asymptotic arguments, not least because such more accurate systems of shell equations tend to be framed in terms of displacements rather than with the help of a stress function.

In the interest of brevity, in this work we have focused exclusively on cylinders with simply supported ends. It is possible to replace those constraints with clamped conditions, as has been done in the numerical studies mentioned at the start of this section. While for the most part there are many similarities between the corresponding problems (with either a uniform or a nonlinear basic state, respectively), some differences exist as well, and these are best discussed separately. For example, as reported by Singer et al. [9], in some of those cases the linear membrane scenario predicts $\Lambda_c > 1$ and a sinusoidal eigenmode, while the nonlinear prebuckling yields $\Lambda_c < 1$ and an edge-localised eigen-deformation (see Fig. 11.28, p. 891, op. cit). Clamping of the circular ends will also introduce small $\mathcal{O}(\alpha^{-1})$ bending layers, whose details are likely to play an important role in piecing together the overall picture. We hope to report the corresponding results at a later time.

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