A novel Group Decision Making method with the prediction selection rate

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Abstract. Methods on the basis of the consensus reaching process are prevalent in Group Decision Making (GDM), which typically forces some evaluators to revise initial opinions in order to reach group consensus without being able to precisely reflect original viewpoints. Furthermore, in case the correct opinion is embedded in the hand of the minority, existing methods may not reach the correct consensus. With the aim to tackle these observations, a novel approach of the Positive and Negative Prediction Selection Rate (PNPSR) is proposed on the basis of the Pythagorean Fuzzy Preference Relation (PFPR) which enables to present individuals' opinions in a pairwise manner using the linguistic preference relation. The PFPR expressed opinions then serve as input for the computation of the proposed PNPSR, the minimum of which is subsequently selected as the correct one. Finally, the full ranking of the alternatives can be calculated through the proposed iterative algorithm. In the process, the evaluators' original opinions are not required to modify, and the correct result can be achieved when the minority evaluators provide the correct opinions. Experimental results demonstrate the efficacy of the proposed approach in comparison with two state-of-the-art methods.

Keywords: Group decision making, Pythagorean fuzzy preference relation, positive and negative prediction selection rate, consensus measure, consensus reaching process

1. Introduction

In human society, when there are important decisions to make, having a committee of experts with different perspectives to vote for a certain motion offers an effective way of decision-making, reducing if not completely avoiding any bias which may otherwise be caused by a single expert [1]. Consequently, a crucial problem in Group Decision Making (GDM) is how to best elicit and aggregate information out of available evaluators. In many scenarios, the simple averaging of all views involving quantity estimation is generally as good as, and often better than, the specific viewpoint resulted from any single evaluator in the group [2]. Despite some evaluators may provide biased opinions, it is crucial for the majority of the evaluators to accept the obtained collective answer [3, 4]. The consensus measure and reaching process for GDM are usually implemented to guide the evaluators to achieve the consensus regarding the derived collective result [5, 6]. In the literature, existing methods to work with the consensus problem for GDM typically include the two following steps:

1) Consensus Measure (CM): It is used in calculating the level of consensus or agreement among the evaluators. Based on input from individual group members, a threshold is typically employed to determine whether the consensus reached can be accepted.

2) Consensus Reaching Process (CRP): When a minority of evaluators hold substantially different opinions from the rest with a low consensus value,

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some original opinions may have to be revised with the feedback or even discarded in case evaluators reject to modify their initial views, with the aim to reach a unified decision from a holistic perspective, such that the final conclusion drawn can be accepted by the majority of the evaluators.

There are a number of different methods proposed in recent literature for CRP [7–10], which generally fall into the strand of developing a revising strategy that guides evaluators with views different from the group majority for the adjustment of initial viewpoints, so that the accepted consensus measure can be achieved with minimum modification of initial views. CRP has achieved successes in numerous scenarios. For example, to the multi-granular unbalanced linguistic information that exists in consensus-based group decision making, zhang et al develop the linguistic computational model to fuse multi-granular unbalanced linguistic terms, then construct the optimization models to generate adjustment advice for evaluators who have to change the opinions in CRP [11]. The individual relations are not regarded for these methods, so a lot of studies have been conducted to the consensus reaching based on social influence evolution in GDM, and the method is also developed to manage the non-cooperative behaviours in group decision-making [12–15]. But these methods have the following potential drawbacks,

1) In CRP, the evaluators have to use certain method to compute the consensus measure with a pre-defined threshold. That is to say, the selection of different computing methods and thresholds could result in potentially different revisions even for the same group decision problem, thus possibly leading to completely different conclusions.

2) In case the minority of the evaluators have to modify original opinions guided by the given advice from CRP, the revised views obviously do not really stand for what these evaluators intend to express originally.

3) In another case where the minority are the true experts whose opinions should have dominated over others, these minority may have to adjust their viewpoints as done in typical CRP processes, which could result in incorrect conclusions.

As a concrete example, students were often asked of the King of Chinese Medicine among highly misleading choices such as Hua Tuo, Sun Simiao and Li Sizhen, all of whom are well-known and had made significant contribution to the domain. The use of conventional GDM methods with consensus measure and CRP could easily lead to the choice of Hua Tuo, which is possibly due to the fact that Hua Tuo was the first person in China utilising anaesthesia in surgery, and more significantly, he had played a impressive role in one of the most well-known historical novels, i.e., the Romance of the Three Kingdoms, making him a household name. However, the choice is incorrect. That is to say, the minority who has made the correct choice of Sun Simiao who is official titled the King of Chinese Medicine, may be dominated by the majority who has the false impression with incorrect knowledge, as the views given by the minority are typically required to modify in CRP, thus leading to incorrect conclusions. The above issues, which have been initially discussed in [16], therefore remain as the research questions to tackle in this paper.

In the general area of GDM, various models and methods have been proposed to work with individual opinions, including the preference representation structures, preference orderings, and preference relations [17]. Particularly, methods developed on the basis of fuzzy theory that has witnessed successes in numerous applications [18] in dealing with uncertainty and imprecision have also been popular in GDM. In general, the evaluators use the fuzzy linguistic information to provide their linguistic assessments. For example, the authors employ the fuzzy linguistic information to conduct the GDM with incomplete information and improve the consistency of fuzzy AHP [19, 20]. The hesitant fuzzy linguistic preference relations can express the uncertainty of evaluation to alternative in the form that is close to people's expression habits, so the evaluation of alternative is closer to the evaluator's real thought [21, 22]. Furthermore, the double hesitant fuzzy linguistic preference relation is introduced to conduct large-scale GDM, in which the authors express the uncertainty of evaluators' cognition to alternative effectively and detailed [23]. In [24], the authors manage the non-cooperative behaviours in large-scale GDM with double hierarchy linguistic preference relation, and the decision making with higher quality can be achieved. In recently, the Intuitionistic Fuzzy linguistic Preference Relation (IFPR) has been introduced to model evaluators' cognitive uncertainty from the positive, negative and hesitant views, which enables to describe the imprecision more accurately than the original fuzzy set [25]. Furthermore, Yager proposes the Pythagorean Fuzzy Set (PFS) [26] which can be considered as an extension of the Intuitionistic Fuzzy Set (IFS) [27], and can deal with more complicated decision-making problems than IFS. The Pythagorean Fuzzy linguistic

Preference Relation (PFPR), on which the proposed approach is based, can therefore be constructed via the PFS [28]. This has led to some relevant studies in the recent literature, such as the one in [29], where the evaluators who have different opinions from the majority are guided to revise their provided answers in order to reach the group consensus.

Inspired by the above observations, this paper proposes a novel group decision making method on the basis of the PFPR. It works by first collecting group members' views using linguistic scale via pairwise comparisons of the alternatives from the positive and negative perspectives. This is followed by converting the obtained results into PFS in order to form the PFPR. The acquired PFPR is then grouped and aggregated from the positive and negative perspectives based on the operators of PFS. This is utilised by the proposed Positive and Negative Predictive Selection Rate (PNPSR) with the aim to select the optimal alternative on top of the previous grouped results. In the above process, CM and CRP are not required, and PNPSR reflects the synthetic difference between the positive and negative views to the alternative. The alternative with the minimal PNPSR is the optimal one, and the true expert has the accurate cognition to the alternative with little uncertainty, therefore the correct result can be obtained when these experts are the minority. Finally, the ranking of the alternatives can be computed by iteratively removing the current best alternative. The advantages of the proposed method can be summarised as follows:

1) The uncertainties embedded in decision making can be captured by constructing the PFPR through the pairwise comparisons of the alternatives from the positive and negative perspectives via Linguistic Discrete Region (LDR).

2) The proposed approach neither requires predefined thresholds for consensus measures, nor forcing evaluators to modify original views, while still guaranteeing eventual consensus.

3) The correct conclusion can still be achieved even when the minority holds the more correct viewpoints while the majority may think otherwise.

4) The rankings of the alternatives are determined through a proposed iterative algorithm, which guarantees the selection of an optimal alternative at each iteration.

The remainder of the paper is organised as follows. In Section 2, the background preliminaries are reviewed. The novel method for GDM is introduced in Section 3. The experimental studies in comparison with two state-of-the-art methods are conducted with discussions provided in Section 4. Section 5 concludes the paper and outlines future work.

2. Preliminaries

In this section, the basic GDM concepts expressed through the Pythagorean Fuzzy Set (PFS) are introduced, including the basic PFS operators and linguistic preference relations, which leads to the idea of the proposed approach.

2.1. Pythagorean fuzzy set

Definition 1. Let X be the universe of discourse. A PFS P in X is described as,

$$P = \{< x, \mu_P(x), \nu_P(x) > |x \in X\} \tag{1}$$

where $\mu_P(x), \nu_P(x) \in [0, 1], 0 \leq (\mu_P(x))^2 + (\nu_P(x))^2 \leq 1, \forall x \in X$. $\mu_P(x)$ and $\nu_P(x)$ are termed the membership and non membership degree of the element x belonging to the P, respectively. The $\pi(x)$ with the definition as follows:

$$\pi(x) = \sqrt{1 - (\mu_A(x))^2 - (\nu_A(x))^2} \tag{2}$$

is termed the degree of indeterminacy of element x belonging to the P. For convenience, $(\mu_P(x), \nu_P(x))$ is named as a Pythagorean Fuzzy Number (PFN) and denoted by $P = (\mu_P, \nu_P)$.

Two basic Pythagorean fuzzy aggregation operators are also presented here, which have been leveraged in the proposed method including the Pythagorean Fuzzy Weighted Averaging (PFWA) and the Pythagorean Fuzzy Weighted Geometric (PFWG) defined as below.

Definition 2. [28] Let $\{p_i = (\mu_i, \nu_i)|i = 1, 2, \ldots, n\}$ be a set of PFNs and $\mathbf{W} = \{\omega_1, \omega_2, \ldots, \omega_n\}^T$ be the weight vector of p_i, with $\sum_{i=1}^{n} \omega_i = 1$, then a PFWA operator is a mapping PA: $P^n \rightarrow P$, where

$$PA(p_1, p_2, \ldots, p_n) = \left(1 - \prod_{i=1}^{n} (1-\mu_i)^{\omega_i}, \prod_{i=1}^{n} (\nu_i)^{\omega_i}\right) \tag{3}$$

Definition 3. [28] Let $\{p_i = \{(\mu_i, \nu_i)|i = 1, 2, \ldots, n\}$ be a set of PFNs and $\mathbf{W} = \{\omega_1, \omega_2, \ldots, \omega_n\}^T$ be the weight vector of p_i, with $\sum_{i=1}^{n} \omega_i = 1$, then a PFWG operator is a mapping

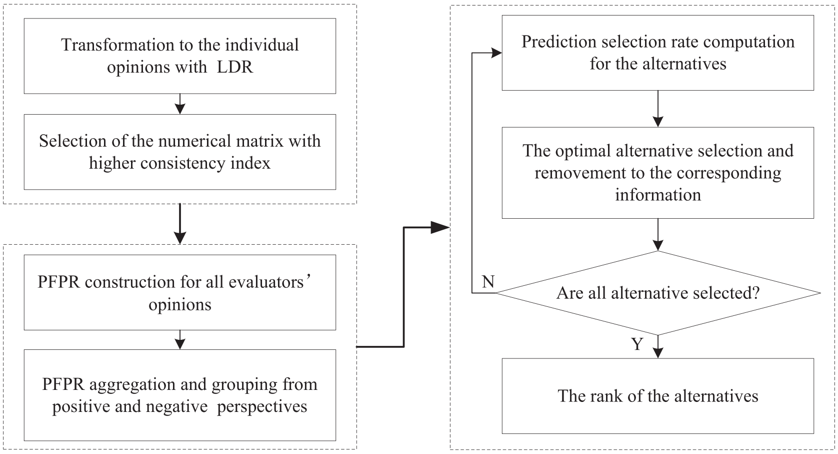

Fig. 1. The flowchart of proposed method.

PG: $P^n \rightarrow P$, where

$$PG(p_1, p_2, \ldots, p_n) = \left(\prod_{i=1}^{n} (\mu_i)^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - \nu_i)^{\omega_i} \right) \quad (4)$$

2.2. Linguistic preference relation

Let $S = \{s_k | k = 0, 1, 2, \ldots, g\}$ be a linguistic term set with the following characteristics:

a) S is ordered: $s_i > s_j$ if and only if $i > j$;

b) There is a negation operator: Neg (s_i) $=s_j$, if $j = g - i$.

The 2-tuple linguistic model is represented as 2-tuples (s_i, α_i), where $s_i \in S$ and $\alpha_i \in [-0.5, 0.5)$ [30]. The linguistic representation model defines a function to perform the transformation between linguistic terms and numerical values.

Let $\beta \in [0, g]$ be a value and the 2-tuple expressing the information equivalent to β is obtained by the following function: $\Delta : [0, g] \rightarrow S \times [-0.5, 0.5)$, $\Delta(\beta) = (s_i, \alpha_i)$, where $i = round(\beta)$ and $\alpha = \beta - i$. Let $\bar{S}$ denote the range of Δ. Δ has an inverse function, $\Delta^{-1} : \bar{S} \rightarrow [0, g]$ and $\Delta^{-1} = (s_i, x) = i + x$.

Let $X = \{x_1, x_2, \ldots, x_n\}(n \geq 2)$ be a finite set of alternatives. When an evaluator makes pairwise comparisons using S, a linguistic preference relation can be constructed as $L = (l_{ij})_{n\times n} \subseteq X \times X$, with a membership function, $u_L : X \times X \rightarrow S$, where $u_L(x_i, x_j) = l_{ij}$ denotes the linguistic preference degree of the alternative x_i over x_j.

3. The proposed method

This section details the steps of the proposed method based on PFPR and presents the flowchart in Fig. 1. This works by first translating the views expressed with LDR from both positive and negative perspectives into numerical set-matrices, through the pairwise comparisons of the alternatives. Then the optimal numerical matrices with higher consistency index are calculated based on the set-matrices from the positive and negative perspectives, for the construction of the PFPR via the optimal matrices and the concept of PFS. The second step aggregates and groups the views which have now been in the form of PFPR, using operators of PFS, with the maximum values acquired from positive and minimum values from negative. Lastly, on top of the previous calculations, an iterative algorithm is proposed to search the correct alternatives and obtain their rankings.

3.1. Constructing initial views in the form of PFPR

As an initial step of the proposed method, the PFPR is constructed once the views expressed by the evaluators have been collected, following the recently proposed approach in [29].

1) The expressions of views given by the evaluators, denoted as $(\mathbb{E} = \{e_k | k \in \{1, 2, \ldots, m\}\})$, are generated based on the pairwise comparisons of the alternatives $(\mathbb{A} = \{A_i | i \in \{1, 2, \ldots, n\}\})$ with LDR, presented using $U_k = (\bar{u}_{ij}^k)_{n\times n}, k \in \{1, 2, \ldots, m\}$ from positive perspectives and $V_k = (\bar{v}_{ij}^k)_{n\times n}, k \in \{1, 2, \ldots, m\}$ from negative viewpoints.

2) The previously formulated U_k and $V_k (k \in \{1, 2, \ldots, m\})$ are then translated into the set-matrices utilizing the 2-tuple linguistic model,

$$r_i = (\sqrt{c})^{\Delta^{-1}(s_i) - \frac{g}{2}} \tag{5}$$

where $c = 2$, and $\Delta^{-1}(s_i) = i (i \in \{0, 1, \ldots, g\})$. Then the iterative search algorithm, as in [31], is used to identify the optimal matrix with highest consistency index from U_k and V_k, $k \in \{1, 2, \ldots, m\}$. Let $O_k^U = (r_{ij(u)}^k)_{n\times n}$ and $O_k^V = (r_{ij(v)}^k)_{n\times n}$ denote the obtained optimal matrices from U_k and V_k, $k \in \{1, 2, \ldots, m\}$.

3) Expressions in the form of PFPR can now be constructed based on the calculated optimal matrix and the concept of PFS. Let $\alpha_{ij}^k = F(r_{ij(u)}^k)$ and $\beta_{ij}^k = F(r_{ij(v)}^k)$ $(i, j \in \{1, 2, \ldots, n\})$ for O_k^U and O_k^V, and the matrix $M_k = \left((\mu_{ij}^k, \nu_{ij}^k)\right)_{n\times n}$, $(k \in \{1, 2, \ldots, m\})$ denotes the constructed PFPR. If $1 - ((\alpha_{ij}^k)^2 + (\beta_{ij}^k)^2) \geq 0$, $(\alpha_{ij}^k, \beta_{ij}^k)$ is a PFN and $(\mu_{ij}^k, \nu_{ij}^k) = (\alpha_{ij}^k, \beta_{ij}^k)$. Otherwise, $(\alpha_{ij}^k - \delta_{ij}^k, \beta_{ij}^k - \delta_{ij}^k)$ is a PFN and $(\mu_{ij}^k, \nu_{ij}^k) = (\alpha_{ij}^k - \delta_{ij}^k, \beta_{ij}^k - \delta_{ij}^k)$, where, $F(r_i) = \frac{1}{2}(1 + \log_{g/2}^{r_i})$ and

$\delta_{ij}^k = \frac{1}{2}\left(\alpha_{ij}^k + \beta_{ij}^k - \sqrt{2 - |\alpha_{ij}^k - \beta_{ij}^k|}\right)$

The following example is used to demonstrate the above mentioned procedure. The iterative searching algorithm can refer to the paper [31].

Example 1. Let "After sale service", "Price of product" and "Brand" be 3 assessment indexes of the cellphone, denote as $\mathbb{A} = \{A_1, A_2, A_3\}$, and the linguistic term set be S = { s_0 = *extremely unimportant*, s_1 = *less unimportant*, s_2 = *slight unimportant*, s_3 = *equally important*, s_4 = *a little important*, s_5 = *more important*, s_6 = *extremely important* }.

The following matrix with LDR is given by a evaluator from the positive (Presenting as U_1) and negative (Presenting as V_1) perspectives, where LDRs denote the uncertainties over the pairwise comparisons of the alternatives:

$$U_1 = \begin{bmatrix} [s_3] & [s_4, s_6] & [s_1, s_2] \\ [s_0, s_2] & [s_3] & [s_1] \\ [s_4, s_5] & [s_5] & [s_3] \end{bmatrix} \text{ and}$$

$$V_1 = \begin{bmatrix} [s_3] & [s_0, s_2] & [s_4, s_5] \\ [s_4, s_6] & [s_3] & [s_5] \\ [s_1, s_2] & [s_1] & [s_3] \end{bmatrix}$$

Using the model $(\sqrt{c})^{\Delta^{-1}(s_k)-3}$ (where $c = 2$), the matrix U_1 is translated into the set-matrix M_s^U,

$$\begin{bmatrix} 1:[1.000] & 3:[1.414, 2.828] & 2:[0.500, 0.707] \\ 3:[0.354, 0.707] & 1:[1.000] & 1:[0.500] \\ 2:[1.414, 2.000] & 1:[2.000] & 1:[1.000] \end{bmatrix}$$

The optimal matrix with higher consistency index can be computed from the set-matrix M_s^U via the iterative search algorithm [31], it is denoted as,

$$A_U^* = \begin{bmatrix} 1.000 & 1.414 & 0.707 \\ 0.707 & 1.000 & 0.500 \\ 1.414 & 2.000 & 1.000 \end{bmatrix}$$

Similarly, the optimal matrix A_V^* is acquired from V_1, as,

$$A_V^* = \begin{bmatrix} 1.000 & 0.500 & 1.414 \\ 2.000 & 1.000 & 2.000 \\ 0.707 & 0.500 & 1.000 \end{bmatrix}$$

Third, the PFPR, denoted as M_1, is constructed by combining the matrices A_U^* and A_V^* via the concept of *PFS*,

$$M_1 = \begin{bmatrix} (0.500, 0.500) & (0.658, 0.185) & (0.342, 0.658) \\ (0.342, 0.816) & (0.500, 0.500) & (0.185, 0.816) \\ (0.658, 0.342) & (0.816, 0.185) & (0.500, 0.500) \end{bmatrix}$$

3.2. Selecting the optimal alternative

With the constructed PFPR matrices $M_k = \left((\mu_{ij}^k, \nu_{ij}^k)\right)_{n\times n}$ $(k \in \{1, 2, \ldots, m\})$, the following result can be obtained based on *PFWG*,

$$P(A_i^k) = \left(\prod_{j=1}^{n} (\mu_{ij}^k)^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \nu_{ij}^k)^{\omega_j}\right) \tag{6}$$

where $\omega_j = \frac{1}{n}$ $(j \in \{1, 2, \ldots, n\})$ is the weight of the alternative A_j. $P(A_i^k)$ is grouped to compute the probability of the alternatives in order to select the optimal one.

Let $P(A_i^k) = (\mu_{A_i}^k, \nu_{A_i}^k)$, if $\max_{A_i}\{\mu_{A_i}^k | A_i \in \mathbb{A}\} = \mu_{A_{i_0}}^k$, $\{\mu_{A_i}^k | A_i \in \mathbb{A}\}$ is added to the group $G_{i_0}^+$. Similarly, if $\min_{A_i}\{\nu_{A_i}^k | A_i \in \mathbb{A}\} = \nu_{A_{i_0}}^k$, $\{\nu_{A_i}^k | A_i \in \mathbb{A}\}$ is added into $G_{i_0}^-$.

This is followed by aggregating $\mu_{A_i}^k$ and $\nu_{A_i}^k$ based on the previously obtained group respectively, where $A_i \in \mathbb{A}$, $k \in \{1, 2, \ldots, m\}$. For group G_j^+ and G_j^-, the numbers of the elements are m_j^+ and m_j^- respec-

Table 1
Degrees of membership of the alternative in group G_1^+

	A_1	A_2	A_3
1	0.7934	0.6839	0.2652
2	0.8338	0.3997	0.4498
3	0.6839	0.3551	0.5172
4	0.8338	0.6070	0.2652
5	0.6839	0.5172	0.3551
6	0.8338	0.3160	0.5172

tively, and $\sum_{j=1}^{n} m_j^+ = m$, $\sum_{j=1}^{n} m_j^- = m$. Let $\mu^l_{A_i(G_j^+)}$ denote the l^{th} degree of membership of alternative A_i in group G_j^+, and $\nu^l_{A_i(G_j^-)}$ be the l^{th} degree of non-membership of alternative A_i in group G_j^-, where $j \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, m_j^+\}$ or $l \in \{1, 2, \ldots, m_j^-\}$. Utilising PFWA, the aggregation for each group and alternative can be presented as,

$$C(G^+_{j(A_i)}) = 1 - \prod_{l=1}^{m_j^+} (1 - \mu^l_{A_i(G_j^+)})^{1/m_j^+} \tag{7}$$

$$C(G^-_{j(A_i)}) = \prod_{l=1}^{m_j^-} (\nu^l_{A_i(G_j^-)})^{1/m_j^-} \tag{8}$$

Furthermore, the probability of the alternative A_i is selected as the optimal alternative in group G_j^+, computed as,

$$p(A_i|G_j^+) = \frac{C(G^+_{j(A_i)})}{\sum_{k=1}^{n} C(G^+_{j(A_k)})} \tag{9}$$

For group G_j^-, the outcome can be computed similarly as,

$$p(A_i|G_j^-) = \frac{C(G^-_{j(A_i)})}{\sum_{k=1}^{n} C(G^-_{j(A_k)})} \tag{10}$$

where $p(A_i|G_j^-)$ denotes the probability of the alternative A_i, which is selected as the optimal alternative in the group G_j^-.

In Example 1, the grouped membership degrees of the alternatives are listed in Table 1 for group G_1^+. The aggregating results of the alternatives in group G_1^+ are calculated via Equation (7) as $\{0.7865, 0.4986, 0.4043\}$. The probability of the alternative A_1 selected as the optimal one in group G_1^+ is calculated as 0.4656, via Equation (9), the probability of the other alternatives can be computed with same way.

Based on the previously selected optimal alternative in the group from the obtained probability of the alternative, the optimal alternative for all evaluators selection can be conducted. Given the assumption that the evaluators who select an alternative as the optimal one from the positive perspective may select the same optimal one from the negative perspective, an important proposition is introduced here, with an associated algorithm proposed to acquire the optimal one and ranking of the alternatives.

Proposition 1. *Let $\mathbb{A} = \{A_1, A_2, \ldots, A_n\}$ be the alternatives, and $\bar{G^+} = \{G_1^+, G_2^+, \ldots, G_n^+\}$ denotes the collection of opinions expressed by individual evaluators from the positive perspective, and $\bar{G^-} = \{G_1^-, G_2^-, \ldots, G_n^-\}$ be the one presented from the negative perspective. Let the probabilities of the alternative A_i selected as the optimal one in group G_j^+ and G_j^- as $P(A_i|G_j^+)$, $P(A_i|G_j^-)$ $(i, j \in \{1, 2, \ldots, n\})$, respectively; then the optimal alternative selected by all evaluators can be presented as A_i^* while $\bar{A}_i^*$ can be used to denote when all evaluators do not select A_i^* as the optimal. It follows that,*

$$\frac{p(G_i^+|A_i^*)}{p(G_i^-|A_i^*)} = \frac{p(A_i^*|G_i^+) \times p(\bar{A}_i^*|G_i^-)}{p(A_i^*|G_i^-) \times p(\bar{A}_i^*|G_i^+)} \tag{11}$$

Assumption, $0/0 \equiv 0$.

Proof. Based on the Bayes theorem, we can obtain, $p(G_i^+|A_i^*) = \frac{p(G_i^+, A_i^*)}{p(A_i^*)}$, $p(G_i^-|A_i^*) = \frac{p(G_i^-, A_i^*)}{p(A_i^*)}$. Dividing the two formulas, it can be obtain,

$$\frac{p(G_i^+|A_i^*)}{p(G_i^-|A_i^*)} = \frac{p(G_i^+, A_i^*)/p(A_i^*)}{p(G_i^-, A_i^*)/p(A_i^*)}$$

$$= \frac{p(G_i^+, A_i^*)}{p(G_i^-, A_i^*)} = \frac{p(A_i^*|G_i^+) \times p(G_i^+)}{p(A_i^*|G_i^-) \times p(G_i^-)}$$

As $p(G_i^+) = \frac{p(G_i^+, \bar{A}_i^*)}{p(\bar{A}_i^*|G_i^+)}$ and $p(G_i^-) = \frac{p(G_i^-, \bar{A}_i^*)}{p(\bar{A}_i^*|G_i^-)}$, thus

$$\frac{p(G_i^+|A_i^*)}{p(G_i^-|A_i^*)} = \frac{p(A_i^*|G_i^+) \times p(G_i^+, \bar{A}_i^*) \times p(\bar{A}_i^*|G_i^-)}{p(A_i^*|G_i^-) \times p(\bar{A}_i^*|G_i^+) \times p(G_i^-, \bar{A}_i^*)}$$

Assuming, $p(G_i^+, A_i^*) = p(G_i^-, A_i^*)$ while A_i^* is the optimal alternative, the following can be derived

$$\frac{p(G_i^+|A_i^*)}{p(G_i^-|A_i^*)} = \frac{p(A_i^*|G_i^+) \times p(\bar{A}_i^*|G_i^-)}{p(A_i^*|G_i^-) \times p(\bar{A}_i^*|G_i^+)}$$

□

Assuming A_i^* is the optimal alternative, let $p(G_i^+|A_i^*)$ (abbreviated as $p(G_i^+)$) denote the probability for the evaluators in group G_i^+ of selecting A_i^*

Table 2
Computed probabilities of alternatives depending on groups

	A_1	A_2	...	A_n	p(G_i)
G_1^+	$p(A_1\|G_1^+)$	$p(A_2\|G_1^+)$	...	$p(A_n\|G_1^+)$	p(G_1^+)
G_1^-	$p(A_1\|G_1^-)$	$p(A_2\|G_1^-)$	...	$p(A_n\|G_1^-)$	p(G_1^-)
G_2^+	$p(A_1\|G_2^+)$	$p(A_2\|G_2^+)$	...	$p(A_n\|G_2^+)$	p(G_2^+)
G_2^-	$p(A_1\|G_2^-)$	$p(A_2\|G_2^-)$	...	$p(A_n\|G_2^-)$	p(G_2^-)
...	...	...	...	...	...
G_n^+	$p(A_1\|G_n^+)$	$p(A_2\|G_n^+)$	...	$p(A_n\|G_n^+)$	p(G_n^+)
G_n^-	$p(A_1\|G_n^-)$	$p(A_2\|G_n^-)$	...	$p(A_n\|G_n^-)$	p(G_n^-)

Table 3
Calculated probabilities of the optimal alternative selection

	A_1	A_2	A_3	$p(G_i)$
G_1^+	0.4655	0.2951	0.2393	0.5000
G_1^-	0.1707	0.3523	0.4769	0.5833
G_2^+	0.3583	0.3979	0.2437	0.1667
G_2^-	0.2732	0.2198	0.5070	0.1667
G_3^+	0.2897	0.3016	0.4087	0.3333
G_3^-	0.3401	0.4069	0.2530	0.2500

as the optimal one from the positive perspective, and $p(G_i^-|A_i^*)$ (abbreviated as $p(G_i^-)$) as the corresponding probability from the negative perspective. The probabilities of the alternative A_i in the group are presented in Table 2. In Proposition 1, it is assumed that the same optimal alternative is selected by evaluators from the positive and negative perspectives. In practice, the probabilities of evaluators selecting the same optimal alternative from both perspectives may vary, and such variation should be as small as possible. As a result, we can infer that $D(A_i, G_i^+, G_i^-)$, which is termed the PNPSR, is the minimum in practice,

$$D(A_i, G_i^+, G_i^-) = \left| \frac{p(G_i^+|A_i^*)}{p(G_i^-|A_i^*)} - \frac{p(A_i^*|G_i^+) \times p(\bar{A}_i^*|G_i^-)}{p(A_i^*|G_i^-) \times p(\bar{A}_i^*|G_i^+)} \right| \tag{12}$$

In Example 1, the obtained probabilities that the evaluators select the optimal alternative in the group and the group rates are shown in Table 3. The process of selecting the optimal alternative can be presented as,

$$\begin{aligned} & D(A_1, G_1^+, G_1^-) \\ & = \left| \frac{0.4167}{0.5000} - \frac{0.4794 \times (0.3758 + 0.4322)}{0.1920 \times (0.2771 + 0.2435)} \right| \\ & = |0.8333 - 3.8765| = 3.3739 \end{aligned}$$

Similarly, $D(A_2, G_2^+, G_2^-) = 1.3466$, $D(A_3, G_3^+, G_3^-) = 0.7079$, so the alternative A_3 is selected as the optimal one. According to Proposition 1 and the above discussions, Algorithm 1 is proposed to obtain the optimal alternative and the pseudo code is presented.

Algorithm 1 Obtaining Optimal One Among Alternatives

Input: matrices $\mathbb{M} = \{M_k = (\mu_{ij}^k, \nu_{ij}^k)_{n\times n} | k \in \{1, 2, \ldots, m\}\}$
alternatives $\mathbb{A} = \{A_i, i \in \{1, 2, \ldots, n\}\}$.
Output: optimal alternative $A_{i_0}^*$
1 Initialize $G_i = \emptyset, i \in \{1, 2, \ldots, n\}$
2 **for** each matrix $M_k = (\mu_{ij}^k, \nu_{ij}^k)_{n\times n} \in \mathbb{M}$ **do**
3 calculate $P_{A_i}^k = (\mu_{A_i}^k, \nu_{A_i}^k)$ based on Equation (6) for each row i
4 $PE_k^U = \{\mu_{A_i}^k | i \in \{1, 2, \ldots n\}\}$
$PE_k^V = \{\nu_{A_i}^k | i \in \{1, 2, \ldots n\}\}$
5 $\mathbb{PE}^U = \{PE_k^U | k \in \{1, 2, \ldots m\}\}^T$
$\mathbb{PE}^V = \{PE_k^V | k \in \{1, 2, \ldots m\}\}^T$
6 **for** each $PE_k^U \in \mathbb{PE}^U$ **do**
7 If $\mu_{Ai_0}^k = \max_i \{\mu_{A_i}^k | i \in \{1, 2, \ldots n\}\}$
then PE_k^U is added into group $G_{i_0}^+$
8 **for** each $PE_k^V \in \mathbb{PE}^V$ **do**
9 If $\nu_{Ai_0}^k = \min_i \{\nu_{A_i}^k | i \in \{1, 2, \ldots n\}\}$
then PE_k^V is added into group $G_{i_0}^-$
10 **for** each group G_j^+ **do**
11 **for** $i = 1$ to n **do**
12 calculate the probabilities $(p(A_i|G_j^+)$ for alternatives according to Equations (7) and (9)
13 **for** each group G_j^- **do**
14 **for** $i = 1$ to n **do**
15 calculate the probabilities $(p(A_i|G_j^-)$ for alternatives according to Equations (8) and (10)
16 **for** each alternative A_i **do**
17 calculate $D(A_i, G_i^+, G_i^-)$ according to Equation (12)
18 $D(A_{i_0}^*, G_{i_0}^+, G_{i_0}^-) = \min_i \{D(A_i, G_i^+, G_i^-)\}$
19 **Return** $A_{i_0}^*$

3.3. Ranking of the alternatives

In Section 3.2, Algorithm 1 is developed based on Proposition 1 for the acquisition of the optimal alternatives, which however does not necessarily return the ranking of the alternatives that can be more informative. In the process of pairwise comparisons of the alternatives, $A_i (i \in \{1, 2, \ldots, n\})$ is compared with the other alternatives $A_k (k \in \{i+1, i+2, \ldots, n\})$ from the positive and negative perspectives. As a result, if the alternative A_j is different from $A_i (i \neq j)$, the pairwise comparisons of A_j with the other alternatives are independent with the comparisons of A_i. When the optimal one, denoted as $A_{i_0}^*$, is selected from the alternatives, the remaining comparisons of other alternatives are independent with the comparisons of A_{i0}^*. Similarly, another optimal one, denoted as $A_{j_0}^* (i_0 \neq j_0)$, can be selected from the remaining alternatives (except A_{i0}^*) based on their pairwise comparisons. That is to say that the process of selecting

Algorithm 2 FOAPN

Input: matrices $\mathbb{U} = \{U_k = (\bar{u}_{ij})_{n\times n} | k \in \{1, 2, \ldots, m\}\}$
and $\mathbb{V} = \{V_k = (\bar{v}_{ij})_{n\times n} | k \in \{1, 2, \ldots, m\}\}$
alternatives $\mathbb{A} = \{A_i, i \in \{1, 2, \ldots, n\}\}$.
Output: Order of alternatives O
Initialize $O = \emptyset$
for $q = 1$ to $n - 1$ **do**
  for each matrix $U_k = (\bar{u}_{ij})_{n\times n}$ **do**
  search the optimal matrix $O_k^U = (r_{ij(u)}^k)_{n\times n}$
  for each matrix $V_k = (\bar{v}_{ij})_{n\times n}$ **do**
  search the optimal matrix $O_k^V = (r_{ij(v)}^k)_{n\times n}$
  construct $\mathbb{M} = \{M_k | k \in \{1, 2, \ldots, m\}\}$
  select the optimal alternative $(A_{i_0}^*)$ from $\mathbb{A}$ according to Algorithm 1 based on $\mathbb{M}$
    $O = O \cup A_{i_0}$
    remove A_{i_0} from $\mathbb{A}$
   for each matrix $U_k = (\bar{u}_{ij})_{n\times n}$ **do**
    calculate U_k^t by removing row i_0 and col i_0 from the matrix U_k
   for each matrix $V_k = (\bar{v}_{ij})_{n\times n}$ **do**
    calculate V_k^t by removing row i_0 and col i_0 from the matrix V_k
    $U_k = U_k^t$, $V_k = V_k^t$
Return O

the optimal one from the alternatives can be iterated with the alternative that has been selected as the optimal one to be removed from the alternatives in current iteration. This selection process is continued until all the alternatives have been selected, and the ranking of the alternatives can thus be obtained as a result of this iterative process.

Formally, Algorithm 2 is developed to compute the ranking of the alternatives based on the above discussions, termed as Finding Orders of Alternatives from Positive and Negative perspectives (FOAPN).

4. Experimental studies

In this section, two case studies are presented to demonstrate the efficacy of the proposed approach in comparison with two state-of-the-art methods.

4.1. Case 1

The first case deals with the scenarios where the evaluators who may have the required knowledge and thus provide correct views are the minority. However, they are not necessarily the most powerful members in the panel who may have the authority without any uncertainty towards the decision problem, but they generally have views for the optimal alternative selection similar from the positive and negative perspectives. While the majority of evaluators may lean towards the incorrect opinions with little uncertainty following on their incorrect impression and/or judgement, and thus could provide views with large differences from the positive and negative perspectives.

The above scenario can be illustrated using a concrete example that has previously been mentioned in the Introduction section, where students are asked to evaluate the possibilities as the King of Chinese Medicine of 3 highly well-known doctors in traditional Chinese medicine, including Hua Tuo, Li Shizhen, Sun Simiao. As a very brief introduction, Hua Tuo is a famous physician from the Han Dynasty of China for being the first person in China to use anaesthesia in surgery, and has become even more widely known for the his impressive role in the popular historical novel, i.e., the Romance of Three Kingdoms. Li Shizhen, from the Ming dynasty of China, is well-known for his major contribution to the writing of Compendium of Materia Medica, a Chinese herbology volume after conducting readings of 800 other medical reference books and carrying out 30 years of field study in the 16^{th} Century. Lastly, Sun Simiao, from the Sui and Tang dynasty, is in fact the correct choice for this question, as he was officially titled as King of Chinese Medicine for his significant contributions to Chinese medicine and tremendous care to patients. It is worth noting that students who are randomly selected for this question generally do not have such precise knowledge of the true fact; whereas only a small minority who are more specialised in traditional Chinese medicine may hold the correct result.

More formally, the three doctors in Chinese medicine, are denoted alternative as $\mathbb{A} = \{A_1, A_2, A_3\}$ for Hua Tuo, Li Shizhen, Sun Simiao, respectively. Twelve students are invited to evaluate the possibility of each one being is King of Chinese Medicine, denoted as the evaluators, $\mathbb{E} = \{e_1, e_2, \ldots, e_{12}\}$. The used linguistic term set is $S = \{s_0 = \textit{extremely impossible}, s_1 = \textit{less impossible}, s_2 = \textit{slightly impossible}, s_3 = \textit{equally possible}, s_4 = \textit{possible}, s_5 = \textit{highly possible}, s_6 = \textit{extremely possible}\}$.

Based on the Bayes theory and Proposition 1, the proposed Algorithm 1 is used to select the optimal alternative. Since there is no consistency problem for the pairwise comparison of the three alternatives, Linguistic Preference Relation (LPR) [3] is utilised in this case for the simplicity. The following subsections illustrate how the proposed method can be used to compute the optimal alternative while the

Table 4
Calculated results based on *PFWG*

No.	$A_1(\mu^k_{A_1}, \nu^k_{A_1})$	$A_2(\mu^k_{A_2}, \nu^k_{A_2})$	$A_3(\mu^k_{A_3}, \nu^k_{A_3})$
1	(0.3161,0.6839)	(0.4828,0.5172)	(0.6449,0.3551)
2	(0.4828,0.4115)	(0.5479,0.5000)	(0.4407,0.6413)
3	(0.5502,0.4498)	(0.6449,0.3160)	(0.1352,0.9289)
4	(0.6840,0.2652)	(0.3884,0.6839)	(0.2066,0.7934)
5	(0.6449,0.2652)	(0.3161,0.8338)	(0.4828,0.6070)
6	(0.5187,0.4813)	(0.3161,0.6839)	(0.6003,0.3997)
7	(0.6840,0.2652)	(0.5187,0.5522)	(0.1352,0.9289)
8	(0.5187,0.4498)	(0.6449,0.3551)	(0.2573,0.8648)
9	(0.3930,0.5778)	(0.3884,0.6413)	(0.6449,0.4115)
10	(0.6449,0.2652)	(0.4828,0.5778)	(0.3161,0.9289)
11	(0.3161,0.5593)	(0.5187,0.4521)	(0.6003,0.5000)
12	(0.6840,0.2652)	(0.1662,0.8115)	(0.4828,0.6413)

Table 5
Probabilities of optimal alternative selection in groups

	A_1	A_2	A_3	$p(G_i)$
G_1^+	0.4794	0.2771	0.2435	0.4167
G_1^-	0.1920	0.3758	0.4322	0.5000
G_2^+	0.3642	0.4325	0.2033	0.2500
G_2^-	0.3039	0.2326	0.4635	0.2500
G_3^+	0.2709	0.2985	0.4307	0.3333
G_3^-	0.3330	0.3876	0.2795	0.2500

other relevant methods with consensus measures and reaching process may not achieve the optimal one.

4.1.1. Experimental results

Once the students provide their views from positive and negative perspectives with LPR, which is then used to construct the *PFPR*, the calculated results based on Equation (6) are shown in Table 4. The results in Table 4 are then grouped based on the strategies discussed in Section 3.2, denoted as $\max_{A_i}\{\mu^k_{A_i}|A_i \in \mathbb{A}\}$ and $\min_{A_i}\{\nu^k_{A_i}|A_i \in \mathbb{A}\}$. Furthermore, the grouped results are aggregated by using *PFWA* which corresponds to Equation (7) and Equation (8), with the probabilities of the optimal alternative selection for each group computed and listed in Table 5.

Based on Equation (12) and the obtained probabilities of the optimal alternative selection in the group, $D(A_i, G_i^+, G_i^-)$ $(i = 1, 2, 3)$ are calculated as $3.0432, 1.5146, 0.6168$ for the alternatives. As a result, the alternative A_3, i.e., Sun Simiao, is selected as the optimal one, consistent with the true fact.

4.1.2. Analysis of the experimental results

The evaluators in groups G_3^+ and G_3^- express views that involve the correct information of the optimal alternative. For instance, the information from e_1 in groups G_3^+ and G_3^- can be presented as the matrix E_1^U and E_1^V, where the i^{th} row denotes the comparison of the alternative A_i over other alternatives for E_1^U from the positive perspective and E_1^V from the negative perspective.

$$E_1^U = \begin{bmatrix} s_3 & s_2 & s_1 \\ s_4 & s_3 & s_2 \\ s_5 & s_4 & s_3 \end{bmatrix} \quad E_1^V = \begin{bmatrix} s_3 & s_4 & s_5 \\ s_2 & s_3 & s_4 \\ s_1 & s_2 & s_3 \end{bmatrix}$$

According to E_1^U, for the comparison of the alternative from the positive perspective, A_1 is compared to A_3, with s_1 suggesting that Hua tuo is "less impossible" to be the King of Chinese Medicine compared with Sun Simiao. A_2 is then used to compare to A_3 with s_2 indicating that Li Sizhen is "slightly impossible" to be the King of Chinese Medicine compared with Sun Simiao. For the comparison of the alternative from negative perspective, A_1 is compared to A_3 with s_5 in E_1^V, which suggests that Hua Tuo is not the King of Chinese Medicine with "highly possible" compared with Sun Simiao.

Furthermore, the student expressed views of A_2 in comparison to A_3, i.e., s_4 in E_1^V, which indicates that Li Sizhen is "possible" not to be the King of Chinese Medicine compared with Sun Simiao. Based on the above analysis, it can be concluded that the answers from e_1 include the information that Sun Simiao has higher possibility of being the King of Chinese Medicine compared with the other two doctors from positive and negative perspectives. For e_6, e_9 and e_{11}, the same results thus can be obtained. Using the views expressed by students in other groups, different decision results can be obtained. For instance, the view expressed by e_4 can be presented as,

$$E_4^U = \begin{bmatrix} s_3 & s_4 & s_6 \\ s_2 & s_3 & s_2 \\ s_0 & s_4 & s_3 \end{bmatrix} \quad E_4^V = \begin{bmatrix} s_3 & s_1 & s_0 \\ s_5 & s_3 & s_4 \\ s_6 & s_2 & s_3 \end{bmatrix}$$

For the comparison of the alternative from the positive perspective, according to E_4^U, when A_1 is compared to A_3, with s_6 ($E_4^U(1, 3)$), suggesting that Hua Tuo is "extremely possible" to be the King of Chinese Medicine compared with Sun Simiao. Comparing A_3 with A_2 results in s_4 ($E_4^U(3, 2)$) that demonstrates Li Sizhen is "possible" to be the King of Chinese Medicine compared with Sun Simiao. Alternatively, comparing A_3 with A_1 results in s_0 ($E_4^U(3, 1)$), suggesting that Sun Simiao is "extremely impossible" to be the King of Chinese Medicine in comparison with Hua Tuo.

For the comparison of the alternative from negative perspective, according to E_4^V, comparing A_1 with A_3

leads to $s_0(E_4^V(1, 3))$, which indicates that Hua Tuo is "extremely impossible" not to be the King of Chinese Medicine compared with Sun Simiao. When A_2 is compared with A_1, the result is $s_5(E_4^V(2, 1))$, which demonstrates that Li Sizhen is "highly possible" not to be the King of Chinese Medicine compared with Hua Tuo. As a result, it can be concluded that Hua Tuo is "less impossible" not to be the King of Chinese Medicine compared with the other two doctors. Based on the above analysis, it is demonstrated that the answers from e_4 include the incorrect information that Hua Tuo has the higher possibility being the King of Chinese Medicine compared with the other two doctors from positive and negative perspectives. Therefore, the above analysis justifies the correct selection made by the proposed method.

4.1.3. Comparison analysis

In the literature, the Intuitionistic Multiplicative Preference Relations (*IMPR*s) and the Intuitionistic Fuzzy Preference Relations (*IFPR*s) can be regarded as the special cases of *PFPR*s, on which the proposed method is based. With a number of research studying the *CRP*s in *GDM* with *IMPR*s [32–34] and *IFPR*s [35–38], this paper compares with two recent methods developed in [35] and [32], since the required data is similar to the acquired one in the experiment, and the comparison can be conducted. It is worth noting that revising information originally provided by evaluators with the unacceptable consensus measure is not advised in [35]. Thus the adapting of *CRP* in [6] is consistent with the proposed method in [35]. Unfortunately, following the procedures described by two state-of-the-art methods, their conclusions suggest that Hua Tuo is the King of Chinese Medicine in the aforementioned example, which is incorrect.

More specifically, for the method reported in [35], the consensuses of evaluations provided by the evaluators are computed as $\{0.7897, 0.8715, 0.7897, 0.8131, 0.8131, 0.8131, 0.7897, 0.8131, 0.8014, 0.7973, 0.8247, 0.8014\}$, with those provided by e_1, e_3, e_7 having the lowest consensus measure. Furthermore, the distribution of original evaluations can be obtained by averaging the distances which are used to compute the consensus measures, denoted as $\{0.1454, 0.0926, 0.1323, 0.1037, 0.1085, 0.1162, 0.1147, 0.1171, 0.1172, 0.1130, 0.1231, 0.1106\}$. It can be observed that those provided by e_1, e_3, e_{11} have larger distances than the others.

Following further on the method in [35], the evaluators e_1, e_3 are suggested to modify their original views in turn. Once the modified evaluations are close to the ones provided by e_4, e_5, e_{10}, e_{12}, belonging to group G_1^+ and G_1^-, their consensus measures are significantly improved. Fox example, the evaluations by e_1 are modified to $E_1^{'}$ as,

$$\begin{bmatrix} (0.5000, 0.5000) & (0.6577, 0.3423) & (0.8155, 0.1845) \\ (0.3423, 0.6577) & (0.5000, 0.5000) & (0.3423, 0.6577) \\ (0.1845, 0.8155) & (0.6577, 0.3423) & (0.5000, 0.5000) \end{bmatrix}$$

which is close to the one provided by e_4, denoted as E_4,

$$\begin{bmatrix} (0.5000, 0.5000) & (0.6577, 0.1845) & (0.9732, 0.0268) \\ (0.3423, 0.8155) & (0.5000, 0.5000) & (0.3423, 0.6577) \\ (0.0268, 0.9732) & (0.6577, 0.3423) & (0.5000, 0.5000) \end{bmatrix}$$

and the consensus measure of $E_1^{'}$ is updated as 0.8598. Based on the modified evaluations, e_1 is added into the group G_1^+ and G_1^-. Then once the completion of *CRP*, the conclusion of Hua Tuo being the King of Chinese Medicine will be drawn.

Alternatively, following on the method proposed in [32], the consensus measures of original evaluations are computed as $\{0.6893, 0.7562, 0.6702, 0.7467, 0.7562, 0.7180, 0.6893, 0.6989, 0.7084, 0.7467, 0.6798, 0.7084\}$, it can be observed that the obtained consensus measures for e_3 and e_{11} are lower than the others. If the original evaluations by e_3 and e_{11} are modified to approach the others, the obtained optimal answers with the accepted consensus measure would have the minimum distance from the evaluations provided by the majority of the evaluators. Based on the revised evaluations, e_1 and e_3 are added into the group G_1^+ and G_1^-. These also lead to the same incorrect conclusion as the one by [35].

4.2. Case 2

The second set of experimental studies is with respect to the evaluation of cellphones, where seven indicators are considered including the $\{$*Memory Size*, *Screen Resolution*, *Brand*, *Camera Pixel*, *After-Sale Service*, *Battery Life*, *Price*$\}$. Each indicator is regarded as an alternative, denoted as $A_i (i \in \{1, 2, \ldots, 7\})$. In order to obtain the ranking of these indexes, 30 evaluators are invited to evaluate the significance of the indexes utilizing the pairwise comparisons from positive and negative perspectives. The linguistic term set utilised in the pairwise comparisons of the indexes is, $\{s_0$: *extremely unimportant*, s_1 : *especially unimportant*, s_2 : *less unimportant*, s_3 : *unimportant*, s_4 : *equally important*, s_5 : *important*,

Table 6
Ranking results of Case 2

Methods	Ranking of Alternatives
Method [35]	$A_4 \succ A_5 \succ A_2 \succ A_6 \succ A_7 \succ A_3 \succ A_1$
Method [32]	$A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_7 \succ A_6$
Proposed	$A_3 \succ A_7 \succ A_5 \succ A_4 \succ A_2 \succ A_6 \succ A_1$

s_6 : *more important*, s_7 : *especially important*, s_8 : *extremely important*}.

4.2.1. Experimental results and comparison analysis

To be consistent with that of Case 1, the two state-of-the-art methods [32, 35] that have been previously used for comparison are again utilised to compare with the proposed method in the second case. Instead of listing the intermediate by-products, which result from following the exact steps as those in Case 1, we directly summarise the final rankings generated by the proposed and two compared methods in Table 6, with the following advantages:

1) The proposed method does not require the determination of weights of the evaluators. As the number of decision maker becomes larger, which could turn to the so-called Large Scale Group Decision Making (LSGDM), and the weight specification will be more difficult especially when methods based on the subjectivity are used. For example, with the 30 evaluators in Case 2, if we have to specify the importance of each individual decision maker, factors such as their experiences and knowledge may have to be accounted for, which would complicate the original problem. On the other hand, if a more objective method is used such as the one recently in [32], there turns out to exist very minor differences in weights between evaluators. For instance, the biggest weight difference using [32] in this case is 0.0012 among 30 evaluators.

2) The proposed method also doesn't require the consensus measure and CRP, nor the modification of evaluations originally provided by the evaluators. Whereas for methods requiring CRP, a threshold is typically specified manually, which would involve another source of subjectivity. Despite the revisions performed by the two compared methods, i.e., [35] and [32] may only involve minor adjustment, the revised evaluations could potentially show notable differences, which does not capture the original perspectives expressed by relevant evaluators. The issue of revising original evaluations may be exaggerated even further in LSGDM, where more evaluators may be forced to adjust their views.

3) For alternative methods using the CRP, once the consensus measure for all evaluators is improved and can be accepted based on revised opinions, the ranking of alternatives is typically generated based on the aggregated matrix, while the selection details for each alternative will not be further utilised. However, the proposed method could exploit the information provided by evaluators more sufficiently by re-calculating the remaining information to compute subsequent optimal ones once an optimal alternative is selected at a certain iteration.

4) The other methods could ignore the evaluations from the negative perspective in CRP. For instance, in CRP, the updated non-member degree of IFPR is calculated via the updated member degree and uncertainty degree for the method in [35], and the no-member degree of IMPR is calculated via the updated member degree in [32]. However, the proposed method could give a more comprehensive consideration by utilising the obtained statistical information to compute the ranking of alternatives and from both the positive and negative perspectives.

5) In the proposed method, if the alternative is good or important, we assume that the obtained evaluating result is good, and the evaluators' cognitions have less uncertainty to this alternative from positive and negative perspectives. As a result, we group the evaluators' opinions according to their optimal selection and aggregate the grouped opinions for every group. According to the aggregated results, we compute the rate that the alternative, such as A3, is selected as the optimal one from positive and negative perspectives, and the rate that this alternative is not selected as the optimal one from the positive and negative perspectives. We analysis more factors in the computation of these rates synthetically, and the result can be observed at the Equation (11) . In the con-

Table 7
The obtained results with the various evaluators and alternatives

	4	5	6	7
15	$\{A_1, A_4, A_2, A_3\}$	$\{A_1, A_2, A_3, A_4, A_5\}$	$\{A_1, A_2, A_6, A_3, A_4, A_5\}$	$\{A_1, A_4, A_6, A_5, A_2, A_7, A_3\}$
20	$\{A_2, A_1, A_3, A_4\}$	$\{A_1, A_2, A_4, A_5, A_3\}$	$\{A_1, A_2, A_4, A_5, A_6, A_3\}$	$\{A_1, A_4, A_5, A_2, A_7, A_6, A_3\}$
25	$\{A_2, A_4, A_1, A_3\}$	$\{A_2, A_1, A_4, A_5, A_3\}$	$\{A_1, A_2, A_3, A_6, A_5, A_4\}$	$\{A_3, A_7, A_1, A_2, A_6, A_5, A_4\}$
30	$\{A_2, A_4, A_1, A_3\}$	$\{A_2, A_1, A_4, A_5, A_3\}$	$\{A_1, A_2, A_3, A_6, A_4, A_5\}$	$\{A_3, A_7, A_5, A_4, A_2, A_6, A_1\}$

ventional method, the difference is directly computed to the result from the positive and negative perspectives. Therefore, the proposed method can obtain the more accurate result than other method. For instance, brand and price are the most important factors to cellphone, the accurate result is achieved by the proposed method.

4.2.2. Experimental results subject to varying numbers of evaluators and alternatives

In order to test the robustness of the proposed method working with different number of evaluators and the alternatives, more experimental scenarios have been set up by varying the number of evaluators in the set $\{15, 20, 25, 30\}$ and the number of alternatives in the set $\{4, 5, 6, 7\}$. The ranking of the alternatives under different scenarios are shown in Table 7. The following observations can be concluded:

1) Under the circumstance where a new alternative is added to existing alternatives that are already ranked, the ranking of the updated collection of alternatives needs to be computed from scratch, as such addition might change the interrelationship between the existing alternatives. For instance, with 20 evaluators and alternatives $\{A_1, A_2, A_3, A_4\}$, its ranking is $\{A_2, A_1, A_3, A_4\}$. However, when an additional A_5 is added into the previous alternatives, the ranking is updated as $\{A_1, A_2, A_4, A_5, A_3\}$, which cannot be achieved by directly inserting the new alternative A_5 into the original ranking of the alternatives.

2) According to Table 7, regardless of the number of the alternatives in most situations, different ranking of the alternatives can be obtained when the number of the evaluators varies, which fits with the common sense that the addition/deletion of experts could potentially affect the overall decision.

As the ranking of the alternatives using the proposed method is computed through the analysis of evaluations provided by evaluators, it is natural the change of the number of evaluators and alternatives could potentially lead to different results. However, in case the number of alternatives is small, the ranking of the alternatives tends to stabilise even when the number of evaluators involved becomes larger. As a result, in practice with *GDM*, selecting the appropriate alternatives and appointing relevant evaluators are significant in getting reasonable results. However, it is generally the case that with more alternatives, it typically requires more evaluators to get involved in order to get a robust result.

4.3. Limitations

We discuss further the potential limitations of the proposed method. In certain situation, the computed PNPSRs of the optimal alternatives are the same or quite close, and the proposed method does not work well. For example, for alternative A_i and A_j $(i \neq j)$, if $D(A_i, G_i^+, G_i^-) = D(A_j, G_j^+, G_j^-)$, and they are the minimum, the optimal alternative can not be selected. In this situation, we can compute,

$$d(A_i, 1) = \left| \frac{p(A_i^*|G_i^+) \times p(\bar{A}_i^*|G_i^-)}{p(A_i^*|G_i^-) \times p(\bar{A}_i^*|G_i^+)} - 1 \right|$$

and

$$d(A_j, 1) = \left| \frac{p(A_j^*|G_j^+) \times p(\bar{A}_j^*|G_j^-)}{p(A_j^*|G_j^-) \times p(\bar{A}_j^*|G_j^+)} - 1 \right|$$

and if $min\{d(A_i, 1), d(A_j, 1)\} = d(A_i, 1)$, A_i is the optimal alternative.

5. Conclusion

Recent GDM methods typically force evaluators to revise original opinions in order to reach the group consensus measure and even so many existing methods may not work well under the circumstance where the correct answer is in the hand of minority. This paper proposes a novel group decision making method on the basis of the PFPR and Bayes theorem. It starts by collecting group members' views using linguistic scale via pairwise comparisons of the alternative from the positive and negative perspectives, which are then converted in the form of PFPR. The acquired PFPR is further grouped and aggregated from the positive and negative perspectives supported by the proposed PNPSR in order to select the optimal alternative. Finally, the ranking of the alternatives can be computed by iteratively removing current best one.

In comparison with two selected state-of-the-art methods, where consensus measures and reaching process are necessary that require to modify original opinions, experimental results demonstrate the efficacy of the proposed method of being able to obtain the correct conclusion even when only a minority of evaluators hold the correct viewpoints. While promising, interesting work remains for further development, including investigating more closely into the theoretical side of the proposed work, applying it more widely into alternative real-world problems and integrating with advanced interpolation techniques [21–24] to deal with scenarios when certain evaluators are not inclined to express or modify any opinions. We will implement the proposed method in the evaluation of tax indicators. In this application, we cannot suggest the few senior tax experts to revise their opinions according to others' assessments, because these invited senior tax experts have rich experience and professionalism.

Further research also includes to combine the proposed method with the observed data and social network evolution to achieve the verifiable result in the application. Therefore, the novel method can be further developed to tackle the challenge that is the minority evaluators can provide the correct views.

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